

**MATH 349**  
**Handout # 2-Solutions.**

1. the series  $\sum_{n=1}^{\infty} \frac{3^n \ln n}{n^n}$

By Ratio test:  $0 < \frac{a_{n+1}}{a_n} = \frac{3^{n+1} \ln(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n \ln n} = \frac{\ln(n+1)}{\ln n} \cdot \frac{3}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$ ,

$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 \cdot 0 \cdot \frac{1}{e} = 0 < 1$ , the series is convergent

(using  $\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} = 1$  by L'H Rule, and  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1}$ ).

2. In a)  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ ; since  $a_n = \frac{1}{n(\ln n)^2} > 0$  for  $n \geq 2$  we can use Integral Test.

The function  $f(x) = \frac{1}{x(\ln x)^2}$  is continuous, positive and decreasing for any  $x \geq 2$

because it is reciprocal of a positive, continuous and increasing (product of 2 incr. funct-s)

function. Now,  $\int \frac{1}{x(\ln x)^2} dx = -\frac{1}{\ln x}$ , using substitution  $u = \ln x, du = \frac{1}{x} dx$ ,

then  $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = -\lim_{x \rightarrow \infty} \frac{1}{\ln x} + \frac{1}{\ln 2} = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ . Therefore the series is convergent.

In b)  $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$  In this case the integral is difficult; try Comparison test:

$\frac{1}{2} \ln n = \ln \sqrt{n} < \sqrt{n}$  for any  $n \geq 2$ ,

so  $\ln n < 2\sqrt{n}$  and  $0 < (\ln n)^2 < 4n$ ,  $\frac{1}{(\ln n)^2} > \frac{1}{4n}$  and harmonic series is divergent

so the given series is divergent.

3.  $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

For  $n \geq 1$ ,  $\frac{1}{n} \leq 1$  and  $0 < \sin \frac{1}{n} < \frac{1}{n}$  since  $\sin x < x$  for  $x > 0$ .

By Comparison Test  $0 < \frac{1}{n} \sin \frac{1}{n} < \frac{1}{n^2}$  and  $p = 2$  so  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent and

$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$  is also convergent.

4.  $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n^n}$  is divergent by Ratio Test :

$$\begin{aligned} 0 < \frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{(n+1)!(n+1)^{n+1}} \cdot \frac{n!n^n}{(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)(n+1)^n} \cdot \frac{n!n^n}{(2n)!} = \\ &= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \cdot \left(\frac{n}{n+1}\right)^n = \frac{2(2n+1)}{n+1} \left(1 - \frac{1}{n+1}\right)^n \rightarrow 2 \cdot 2 \cdot e^{-1} = \frac{4}{e} > 1 \end{aligned}$$

5.  $\sum_{n=1}^{\infty} \frac{e^n \cos^2 n}{\pi^n - 1}$

Since  $0 < \cos^2 n < 1, 0 < \frac{e^n \cos^2 n}{\pi^n - 1} \leq \frac{e^n}{\pi^n - 1} = a_n$ , but this sequence is equivalent

to the geometric sequence  $b_n = \frac{e^n}{\pi^n}$ , where  $r = \frac{e}{\pi} < 1$

since  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{e^n}{\pi^n - 1} \cdot \frac{\pi^n}{e^n} = \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{\pi^n}} = 1$ .

By Limit Comparison Test  $\sum_{n=1}^{\infty} a_n$  is convergent

and by Comparison Test the original series is convergent.

6. Find the sum of  $\sum_{n=2}^{\infty} \frac{1}{e^{\frac{n}{2}}}$

the series is a geometric one where  $r = \frac{1}{\sqrt{e}} < 1$  so the series is convergent

and

$$\sum_{n=2}^{\infty} \left(\frac{1}{e^{\frac{1}{2}}}\right)^n = \frac{\left(\frac{1}{\sqrt{e}}\right)^2}{1 - \frac{1}{\sqrt{e}}} = \frac{1}{e} \cdot \frac{\sqrt{e}}{\sqrt{e} - 1} = \frac{1}{\sqrt{e}(\sqrt{e} - 1)}$$

using  $\sum_{n=N}^{\infty} r^n = \frac{r^N}{1 - r}$  for any  $-1 < r < 1$ .

7. the series  $\sum_{n=1}^{\infty} \frac{2 + \cos n}{\sqrt{n} + n}$  has positive terms and  $1 \leq 2 + \cos n \leq 3$

Also  $2 \cdot \sqrt{n} \leq \sqrt{n} + n \leq 2n$

so  $\frac{1}{2n} \leq \frac{2 + \cos n}{\sqrt{n} + n} \leq \frac{3}{2\sqrt{n}}$  and we can use the left part of the inequality

and Comparison test. Since  $\sum_{n=1}^{\infty} \frac{1}{2n}$  is divergent

(a half of the harmonic series) and our series is bigger it is also divergent.

8. the series  $\sum_{n=1}^{\infty} \frac{5^n}{n^{n+1}}$  has positive terms so we can try Ratio or Root test

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)^{n+2}} \cdot \frac{(n)^{n+1}}{5^n} = \frac{5}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^{n+1} = \frac{5}{n+1} \cdot \left(1 - \frac{1}{n+1}\right)^{n+1} \rightarrow 0 \cdot e^{-1} = 0$$

as  $n \rightarrow \infty$ . Since the limit  $\rho = 0 < 1$  the series is convergent.

Root test is easier

$$(a_n)^{\frac{1}{n}} = \left(\frac{5^n}{n^{n+1}}\right)^{\frac{1}{n}} = \frac{5}{n^{1+\frac{1}{n}}} \rightarrow \frac{5}{\infty} = 0 \text{ as } n \rightarrow \infty \text{ since } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

Since the limit  $\sigma = 0 < 1$  the series is convergent.

9. Find the sum of  $\sum_{n=1}^{\infty} \frac{5 + 2^n}{5^{n+2}}$ .

We can split the series into two convergent ones:

$$\sum_{n=1}^{\infty} \frac{5 + 2^n}{5^{n+2}} = \sum_{n=1}^{\infty} \frac{5}{5^{n+2}} + \sum_{n=1}^{\infty} \frac{2^n}{5^{n+2}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{5^n} + \frac{1}{5^2} \sum_{n=1}^{\infty} \frac{2^n}{5^n}$$

both are convergent geometric series with  $r = \frac{1}{5}$  and  $r = \frac{2}{5}$  so the original series is convergent and the sum

$$s = \frac{1}{5} \cdot \frac{\frac{1}{5}}{1 - \frac{1}{5}} + \frac{1}{25} \cdot \frac{\frac{2}{5}}{1 - \frac{2}{5}} = \frac{1}{20} + \frac{2}{75} = \frac{23}{300}$$

using  $\sum_{n=N}^{\infty} r^n = \frac{r^N}{1-r}$  for any  $-1 < r < 1$ .

10. the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  has positive terms for  $n \geq 2$

the function  $f(x) = \frac{\ln x}{x}$  is positive and continuous on  $[2, \infty)$ , and

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \text{ for } x > e \text{ since } \ln x > \ln e = 1,$$

so the function is decreasing on  $[3, \infty)$  and we can use Integral test :

$\sum_{n=3}^{\infty} \frac{\ln n}{n}$  is divergent since the integral

$$\int_3^{\infty} \frac{\ln x}{x} dx = \int_{\ln 3}^{\infty} u du \text{ (by subst. } u = \ln x, du = \frac{dx}{x}) = \left[ \frac{u^2}{2} \right]_{\ln 3}^{\infty} = \infty.$$

11. the series  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$  has positive terms so we can try Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2 \cdot (n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \rightarrow \frac{1}{4}$$

as  $n \rightarrow \infty$  (divide the top and bottom by  $n^2$ ). Since the limit  $\rho = \frac{1}{4} < 1$

the series is convergent.