

The University of Calgary  
 Department of Mathematics and Statistics  
 MATH 349  
 Handout # 3 Solution

**For 1a)**

$\sum_{n=3}^{\infty} |\dots| = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)}$  is divergent by Integral test:

$f(x) = \frac{1}{x \ln x}$  is positive, continuous and decreasing for  $x > 1$

since  $x \ln x$  is the product of two positive and increasing functions,

The integral is divergent, since

$$\int_3^{\infty} \frac{1}{x(\ln x)} dx = \lim_{x \rightarrow \infty} \ln(\ln x) - \ln \ln 3 = \infty$$

so the original series is absolutely divergent.

**For 1b)**

it is conditionally convergent since  $a_n = \frac{1}{n(\ln n)} \rightarrow 0 \left(\frac{1}{\infty}\right)$

and the sequence is decreasing — see above  $f$  is decreasing function

**For 2a)**

the series  $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$  is divergent by Comparison Test since  $0 < \ln x < x$  for  $x > 1$ ,

$$\ln(n+1) < n+1 \quad \frac{1}{\ln(n+1)} > \frac{1}{n+1}$$

and  $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{k=2}^{\infty} \frac{1}{k}$  which is divergent harmonic series (by Integral Test).

**For 2 b)**

the series is conditionally convergent by Alternating Test since the sequence

$a_n = \frac{1}{\ln(n+1)}$  has positive terms, limit 0  $\left(\frac{1}{\infty}\right)$  and is decreasing

since  $\ln x$  is increasing, positive for  $x > 1$ , and " $\frac{1}{\text{incr, pos}}$ " = decr.

**For 3a)**

investigate  $\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n!}\right)$  since  $a_n = \left(\frac{1}{n} - \frac{1}{n!}\right) > 0$  for  $n \geq 3$  We can split into two series

harmonic series  $\sum_{n=2}^{\infty} \frac{1}{n}$  which is divergent and the series  $\sum_{n=2}^{\infty} \frac{1}{n!}$  which is convergent

by Ratio Test:  $0 < \frac{1}{(n+1)!} \cdot n! = \frac{1}{n+1} \rightarrow 0 < 1$ ,

so together the series  $\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n!}\right)$  is divergent. The original series is absolutely divergent.

Also by Comparison Test for  $n \geq 3$   $n! \geq 2n$  so  $\frac{1}{n!} \leq \frac{1}{2n}$  and

$a_n \geq \frac{1}{2n}$  for  $n > 2$ .

**For 3b)**

we can separate again since  $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n}$  is conditionally convergent by Alternating Test:

the sequence  $\left\{\frac{1}{n}\right\}$  is decreasing, with positive terms and limit 0.

And the second series  $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n!}$  is absolutely convergent from above ,

so together the original series is **conditionally convergent**.

Also directly by Alternating test:

the sequence  $a_n$  from above has limit 0 ,so we have to show that it is decreasing:

$$a_{n+1} < a_n \quad \frac{1}{n+1} - \frac{1}{(n+1)!} < \frac{1}{n} - \frac{1}{n!}, \text{ multiply both sides by } (n+1)! :$$

$$n! - 1 < (n+1)(n-1)! - (n+1), \text{ so } n! < n(n-1)! + (n-1)! - n, 0 < (n-1)! - n$$

finally  $n < (n-1)(n-2) < (n-1)!$  which is true for  $n \geq 4$

**For 4a)**

$$\text{the centre is } c = -1 \text{ and } a_n = \frac{n!}{4^n}, \text{ so } 0 < \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{4^{n+1}} \cdot \frac{4^n}{n!} = \frac{n+1}{4} \rightarrow \infty,$$

so  $R = \frac{1}{L} = 0$  and the series converges ONLY for  $x = -1$ .

**For 4b)**

$$\begin{aligned} \text{the centre is } c = \frac{1}{2} \text{ and } a_n &= \frac{n! \cdot 2^n}{(2n)!}, \text{ so } 0 < \frac{a_{n+1}}{a_n} = \frac{(n+1)!2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!2^n} = \\ &= \frac{(n+1)2}{(2n+2)(2n+1)} = \frac{1}{2n+1} \rightarrow 0 \text{ so } R = \frac{1}{L} = +\infty, \text{ and the interval is } (-\infty, +\infty). \end{aligned}$$

**For 5)**

The answer must be in the form  $\sum_{n=1}^{\infty} a_n (x-1)^n$ . We know that for geometric series

$$\sum_{n=0}^{\infty} (-1)^n r^n = \frac{1}{1+r} \text{ for any } -1 < r < 1.$$

$$\text{So first } \frac{1}{x+1} = \frac{1}{(x-1)+2} = \frac{1}{2} \cdot \frac{1}{1+\frac{x-1}{2}} \quad \left( r = \frac{x-1}{2} \right)$$

$$\frac{1}{x+1} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^n} \text{ for } -1 < x < 3 \text{ (from } -1 < \frac{x-1}{2} < 1).$$

$$\text{Differentiate both sides: } \frac{-1}{(x+1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \cdot n(x-1)^{n-1}, (n-1 = k)$$

$$\frac{1}{(x+1)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{2^{k+2}} (x-1)^k \quad \text{for } -1 < x < 3.$$

**For 6a)**

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n (x-\frac{1}{4})^n}{n^n} \quad \text{the centre is } c = \frac{1}{4} \text{ and } a_n = \frac{4^n}{n^n}.$$

$$\text{Since } \left| \frac{a_{n+1}}{a_n} \right| = \frac{4^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{4^n} = \frac{4}{n+1} \cdot \left( \frac{n}{n+1} \right)^n \rightarrow L = 0 \cdot \frac{1}{e} = 0,$$

$$\text{using } \lim_{n \rightarrow \infty} \left( 1 + \frac{a}{n} \right)^n = e^a \quad R = \frac{1}{L} = +\infty, \text{ the interval is } (-\infty, +\infty).$$

Easier by Root Test  $(|a_n|)^{\frac{1}{n}} = \frac{4}{n} \rightarrow 0$ .

**For 6b)**

$$\text{the centre is } c = 4 \text{ and } a_n = (-1)^n \frac{n}{2^n}, \text{ since } \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \rightarrow \frac{1}{2}$$

the radius is  $R = 2$  and series is absolutely convergent on  $]2, 6[$ .

Now, for  $x = 2$

we have to investigate  $\sum_{n=1}^{\infty} n = +\infty$  and for  $x = 6$  the series  $\sum_{n=1}^{\infty} (-1)^n n$  which is also divergent since the limit of the  $n$ -th term is NOT 0.

So the interval is open  $]2, 6[$ .

**For 7)**

The answer must be in the form  $\sum_{n=0}^{\infty} a_n (x+1)^n$ .

Rewrite  $\ln(2-x) = \ln(3-(x+1)) = \ln\left[3\left(1-\frac{x+1}{3}\right)\right] =$   
 $= \ln 3 + \ln\left(1-\frac{x+1}{3}\right) = \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n3^n} (x+1)^n$  for  $x \in [-4, 2[$

since  $-1 \leq \frac{x+1}{3} < 1$        $-3 \leq x+1 < 3$        $-4 \leq x < 2$

using  $\ln(1-s) = -\sum_{n=1}^{\infty} \frac{1}{n} s^n$  for  $s \in [-1, 1[$ .

**For 8)**

The answer must be in the form  $\sum_{n=0}^{\infty} a_n (x+4)^n$ .

$$\frac{1}{1-2x} = \frac{1}{1-2(x+4-4)} = \frac{1}{9-2(x+4)} = \frac{1}{9} \cdot \frac{1}{1-\frac{2}{9}(x+4)}$$

now, using  $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$  for  $-1 < r < 1$ , where  $r = \frac{2(x+4)}{9}$  we get

$$\frac{1}{1-2x} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{2^n}{9^n} (x+4)^n = \sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}} (x+4)^n, \text{ so } a_n = \frac{2^n}{9^{n+1}}$$

Now, to find the interval, solve for  $x$ :  $-1 < \frac{2(x+4)}{9} < 1$

$$-9 < 2x+8 < 9 \quad \frac{-17}{2} < x < \frac{1}{2}.$$