

**MATH 349**  
**Handout #4      Solution**

**For 1)**

$$f(x, y) = \sqrt{2x + y^2}$$

for the domain  $2x + y^2 \geq 0$        $y^2 \geq -2x$       any  $x > 0$  will do,

generally the region right of the parabola  $y^2 = -2x$ ,  $x \leq 0$

level curves:  $c = 0$        $0 = 2x + y^2$ ... parabola above;

$c < 0$ ... NO curves;  $c > 0$ ...parabolas shifted to the right

$$y^2 = c^2 - 2x \quad (y = \pm\sqrt{c^2 - 2x} \quad \text{the range is } [0, +\infty)$$

Partials:

$$f_x = \frac{2}{2\sqrt{2x + y^2}} = \frac{1}{\sqrt{2x + y^2}}; f_y = \frac{2y}{2\sqrt{2x + y^2}} = \frac{y}{\sqrt{2x + y^2}}$$

obviously  $y f_x = f_y$  in the domain except on  $y^2 = -2x$ .

**For 2)**

$$\text{For } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + 2y^2}$$

define  $g(x, y) = \frac{x^3}{x^2 + 2y^2}$  then  $g(0, y) = 0$  for  $y \neq 0$  and for  $x \neq 0$

$g(x, 0) = \frac{x^3}{x^2} = x \rightarrow 0$  as  $x \rightarrow 0$ ; along any line  $y = mx$  for  $x \neq 0$

$$g(x, mx) = \frac{x^3}{x^2 + 2m^2x^2} = \frac{x}{1 + 2m^2} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Thus limit could be 0. Now, we have to prove it

$$|g(x, y) - 0| = \left| \frac{x^3}{x^2 + 2y^2} \right| = |x| \frac{x^2}{x^2 + 2y^2} \leq |x| \frac{x^2 + 2y^2}{x^2 + 2y^2} = |x| \rightarrow 0$$

as  $x \rightarrow 0$  therefore  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + 2y^2} = 0$  by Squeeze Theorem.

**For 3)**

For  $f(x, y) = \frac{e^y}{x}$  the domain is  $x \neq 0$ , so  $D = \mathbf{R}^2 - \{y\text{-axis}\}$

level curves:  $c = 0$ , NO curve since  $e^y > 0$  always

for  $c \neq 0$        $cx = e^y$ ,  $\ln cx = y$ , for  $cx > 0$ , and  $y = \ln |x| + \ln |c|$

so for  $c = 1$ ,  $y = \ln x$ ,  $x > 0$ ; for  $c = -1$ ,  $y = \ln(-x)$ ,  $x < 0$ ,

and for  $c = e$ ,  $y = \ln x + 1$ ,  $x > 0$

the range is  $(-\infty, 0) \cup (0, +\infty)$

For the derivative  $f_x = -\frac{e^y}{x^2}$ , and  $(f_x)_y = -\frac{e^y}{x^2} = f_{xy}$ ,

since all functions are continuous in  $D$        $f_{xy} = f_{yx}$ .

**For 4)**

for  $f(x, y) = \left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{1}{2}}$

using Chain Rule  $f_x = -\frac{1}{2} \left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (2xy + x^3)$  and

$$f_y = -\frac{1}{2} \left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (x^2 + 2y), \text{ so } xf_y = f_x.$$

For the domain:  $yx^2 + y^2 + \frac{1}{4}x^4 > 0$ ,  $\left(y + \frac{1}{2}x^2\right)^2 > 0$ ,

always except when  $y = -\frac{1}{2}x^2$

Also we can simplified  $f(x, y) = \left|y + \frac{1}{2}x^2\right|^{-1}$ , so  $f_x = \frac{-x}{(y + \frac{1}{2}x^2)^2} \operatorname{sgn}\left(y + \frac{1}{2}x^2\right)$ ,

and  $f_y = \frac{-1}{(y + \frac{1}{2}x^2)^2} \operatorname{sgn}\left(y + \frac{1}{2}x^2\right)$ , if  $y + \frac{1}{2}x^2 \neq 0$ .

**For 5)**

For  $f(x, y) = \ln(x^2 + y^2 + x)$

for domain solve  $x^2 + y^2 + x > 0$  complete the square:  $\left(x + \frac{1}{2}\right)^2 + y^2 > \frac{1}{4}$

so the domain is outside the circle with the centre at  $\left(-\frac{1}{2}, 0\right)$  and  $r = \frac{1}{2}$ .

For the level curves:  $e^c = x^2 + y^2 + x$  so as above :

$$\left(x + \frac{1}{2}\right)^2 + y^2 = e^c + \frac{1}{4}$$

so all are circles with the same centre  $\left(-\frac{1}{2}, 0\right)$  and  $r = \sqrt{e^c + \frac{1}{4}}$ ,

for any  $c$  thus the range is  $(-\infty, \infty)$

particularly if  $c = 0$   $r = \frac{\sqrt{5}}{2}$ ;  $c = \ln 2$   $r = \frac{3}{2}$ ;

$c = -\ln 2 = \ln \frac{1}{2}$   $r = \frac{\sqrt{3}}{2}$ ...

Notice that cross-section  $x = 0$   $z = 2 \ln |y|$

or for  $y = 0$   $z = \ln(x^2 + x)$

**For 6)**

define  $g(x, y) = \frac{xy + y}{(x + 1)^2 + y^2} = \frac{(x + 1)y}{(x + 1)^2 + y^2}$  for  $(x, y) \neq (-1, 0)$

then  $g(-1, y) = \frac{0}{y^2} = 0$  for  $y \neq 0$  and  $g(x, 0) = 0$  for  $x \neq -1$

but for the line  $y = x + 1$  through the point  $(-1, 0)$

$g(x, x + 1) = \frac{(x + 1)^2}{2(x + 1)^2} = \frac{1}{2}$  for  $x \neq -1$  so the limit DNE.

**For 7)**

$f(x, y) = \arctan \frac{x}{y}$  for  $y \neq 0$

$$f_x = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2} \text{ and } f_y = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} = \frac{-x}{x^2 + y^2}$$

so  $y f_x - x f_y = \frac{y^2 + x^2}{x^2 + y^2} = 1$  for any  $x$  and  $y \neq 0$ .

**For 8)**

For  $f(x, y) = \frac{2x}{x^2 + y}$  the domain is

$$D = \{y \neq -x^2\} \dots xy\text{-plane except the parabola } y = -x^2$$

level curves  $c = 0 \implies 2x = 0, x = 0 \dots y\text{-axis except the origin}$

for  $c = 1$ :  $1 = \frac{2x}{x^2 + y}, x^2 + y = 2x, y = -x^2 + 2x = -x(x - 2)$

.a parabola open down, vertex  $V(1, 1)$ , roots 0, 2 without the origin

for  $c = -2$ :  $-2 = \frac{2x}{x^2 + y}, -2x^2 - 2y = 2x, y = -x(1 + x)$

a parabola open down, vertex at  $V\left(-\frac{1}{2}, \frac{1}{4}\right)$ , roots 0, -1 without the origin

general level curves are shifted parabolas open down without the origin,

passing through the origin

now, partials

$$f_x(x, y) = 2 \cdot \frac{x^2 + y - 2x^2}{(x^2 + y)^2} = \frac{2(y - x^2)}{(x^2 + y)^2} \text{ at } x = -1, y = 1 \quad f_x(-1, 1) = 0$$

$$f_y(x, y) = 2x \cdot (-1)(x^2 + y)^{-2} \cdot 1 = \frac{-2x}{(x^2 + y)^2} \text{ at } x = -1, y = 1 \quad f_y(-1, 1) = \frac{2}{4} = \frac{1}{2}$$

so a normal vector to the tangent plane is  $(f_x, f_y, -1) = (0, \frac{1}{2}, -1)$

or  $\vec{n} = (0, 1, -2)$  an equation is  $y - 2z = d$

for  $d$  we need the point  $z_0 = f(-1, 1) = \frac{-2}{2} = -1$   $P(-1, 1, -1)$

and  $1 + 2 = 3 = d$ , so together  $y - 2z = 3$ .

**For 9)**

define  $g(x, y) = \frac{x(y-1)}{3x^2 + 2(y-1)^4}$  for  $(x, y) \neq (0, 1)$

then for  $y \neq 1$   $g(0, y) = 0$  and for  $x \neq 0$   $g(x, 1) = 0$ ,

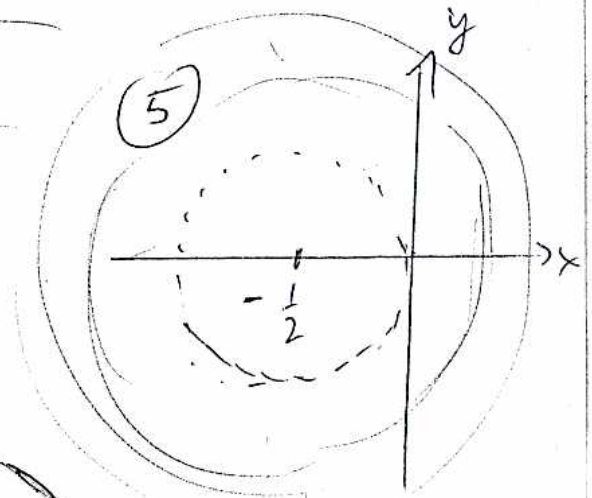
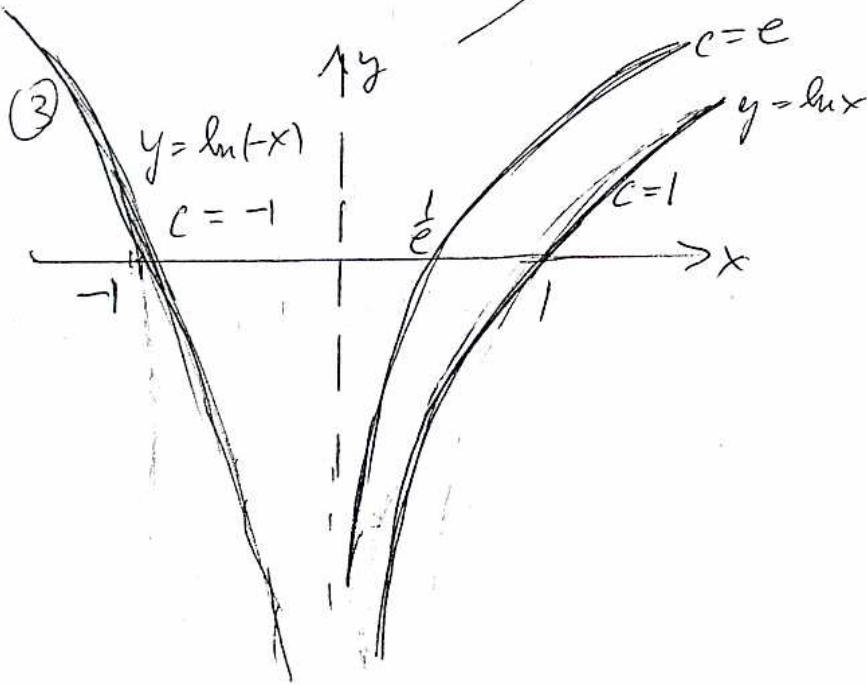
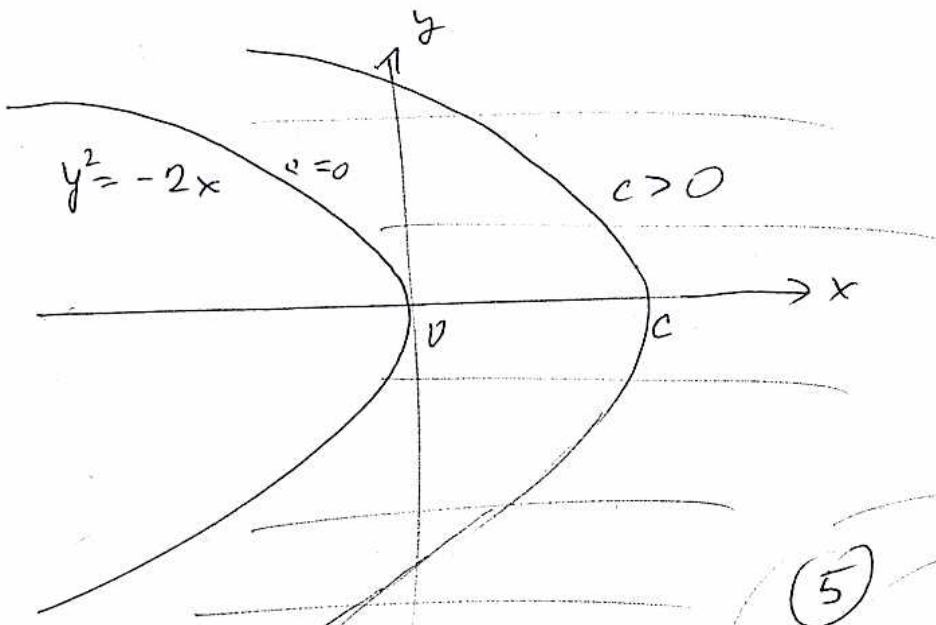
try a line through that point  $y - 1 = mx, m \neq 0$

for  $x \neq 0$   $f(x, mx + 1) = \frac{mx^2}{3x^2 + 2m^4x^4} = \frac{m}{3 + 2m^4x^2} \rightarrow \frac{m}{3} \neq 0$

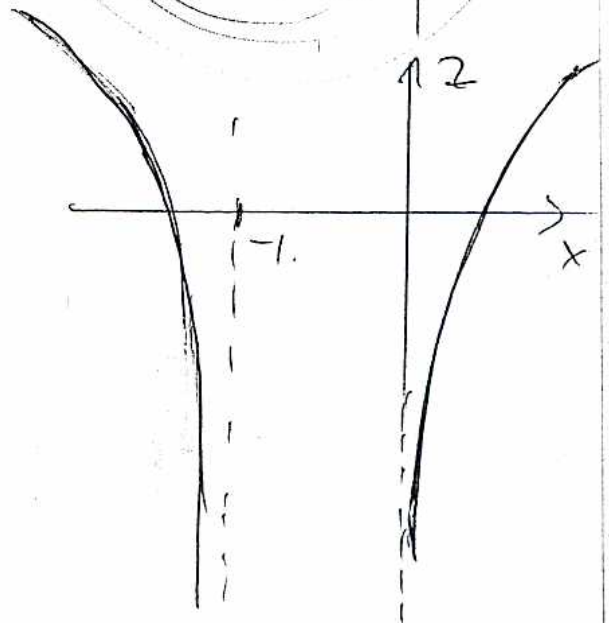
for any  $m \neq 0$  (as  $x \rightarrow 0$ )

Therefore the limit DNE.

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