The University of Calgary
Department of Mathematics and Statistics
MATH 349 Handout # 5

Solutions **For 1 a)**

For $f(x,y) = \dfrac{xy}{\sqrt{1+x^2}}$ $\quad f_y = \dfrac{x}{\sqrt{1+x^2}}$ and

$$f_x = y \cdot \frac{\sqrt{1+x^2} - x\frac{2x}{2\sqrt{1+x^2}}}{(1+x^2)} = y\frac{(1+x^2)-x^2}{\sqrt{1+x^2}\,(1+x^2)} = \frac{y}{(1+x^2)^{\frac{3}{2}}}$$

For 1b)

for $x = \cos(\pi st)$ $\quad y = \sin\dfrac{\pi s}{t}$ $\quad \dfrac{\partial x}{\partial s} = -\sin(\pi st)\cdot \pi t$ $\quad \dfrac{\partial y}{\partial s} = \cos\dfrac{\pi s}{t}\cdot\dfrac{\pi}{t}$

and $\dfrac{\partial x}{\partial s}(0,-1) = 0$ $\quad \dfrac{\partial y}{\partial s}(0,-1) = -\pi$

for $s=0$ and $t=-1$ $\quad x = \cos(0) = 1, y = \sin 0 = 0$

and $\nabla f(1,0) = \left(0, \dfrac{1}{\sqrt{2}}\right)$

$\frac{\partial}{\partial s}h(0,-1) = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s} = \nabla f(1,0) \bullet \left(\frac{\partial x}{\partial s}(0,-1), \frac{\partial y}{\partial s}(0,-1)\right) =$

$= \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0,-\pi) = \dfrac{-\pi}{\sqrt{2}}$

$\dfrac{\partial x}{\partial t} = -\sin(\pi st)\cdot \pi s$ $\quad \dfrac{\partial y}{\partial t} = \cos\dfrac{\pi s}{t}\cdot\dfrac{-\pi s}{t^2}$ and $\dfrac{\partial x}{\partial t}(0,-1) = 0$ $\quad \dfrac{\partial y}{\partial t}(0,-1) = 0$

$\frac{\partial}{\partial t}h(0,-1) = f_x\frac{\partial x}{\partial t} + f_y\frac{\partial y}{\partial t} = \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0,0) = 0$

OR by Chain Rule $\quad Dh = Df\,D\mathbf{g}$ $\quad$ matrix multiplication where

$Df = [f_x \quad f_y] = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}$ at $x=1, y=0$

$\mathbf{Dg} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$ at $s=0, t=-1$ $\quad \mathbf{Dg}(0,-1) = \begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix}$

so $Dh = [h_s \quad h_t] = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{-\pi}{\sqrt{2}} & 0 \end{bmatrix}$

For 2)

for $f(x,y) = \ln(x+y^2)$ and $x_0 = 0, y_0 = -1$ $\quad z_0 = f(0,-1) = \ln 1 = 0$

partials $f_x = \dfrac{1}{x+y^2}$ $\quad f_y = \dfrac{2y}{x+y^2}$ and $A = f_x(0,-1) = 1, B = f_y(0,-1) = -2$

so tangent plane $\quad z = z_0 + A(x-x_0) + B(y-y_0)$

$z = 0 + (x-0) - 2(y+1)$ $\quad z = x - 2y - 2$

Or the point on the graph is $P(0,-1,0)$ and $\mathbf{n} = (\nabla f, -1) = (1,-2,-1)$

so $x - 2y - z = d$ and from P $\quad 0 + 2 - 0 = d$ thus $x - 2y - z = 2$.

For 3)

for $f(x,y) = e^{\sqrt{\frac{y}{x}}}$ the domain of f is $\frac{y}{x} \geq 0$ i.e.$\{y \geq 0, x > 0\} \cup \{y \leq 0, x < 0\}$

but the domain of the partials is .$\{y > 0, x > 0\} \cup \{y < 0, x < 0\} = \{xy > 0\}$

$f_x = e^{\sqrt{\frac{y}{x}}} \cdot \frac{1}{2}\left(\frac{y}{x}\right)^{-\frac{1}{2}} \cdot \frac{-y}{x^2} = -\frac{\sqrt{y}}{2x^{\frac{3}{2}}}e^{\sqrt{\frac{y}{x}}}$ for $x > 0, y > 0$ only since $\sqrt{\dfrac{y}{x}} = \dfrac{\sqrt{y}}{\sqrt{x}}$

$f_{xx} = e^{\sqrt{\frac{y}{x}}}\left[\left(-\frac{\sqrt{y}}{2}x^{-\frac{3}{2}}\right)^2 + \frac{3}{4}\sqrt{y}x^{-\frac{5}{2}}\right] = e^{\sqrt{\frac{y}{x}}}\left[\dfrac{y}{4x^3} + \dfrac{3}{4x^2}\sqrt{\dfrac{y}{x}}\right]$

$f_{xy} = -\frac{1}{2}x^{-\frac{3}{2}}e^{\sqrt{\frac{y}{x}}}\left[\frac{1}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{y}\sqrt{x}}\right] = -\frac{1}{4}e^{\sqrt{\frac{y}{x}}} \cdot \left(\frac{1}{x\sqrt{yx}} + \frac{1}{x^2}\right)$, for $xy > 0$
the middle steps are true only if both x, y are positive
but the final expresions are valid even if both are negative.

for $z > 0$ $\quad \nabla f = gradf = (f_x, f_y, f_z)$ where
$f_x = \sqrt{2}\cos(\pi xy + x\ln z)\,[\pi y + \ln z]$ $\quad f_y = \sqrt{2}\cos(\pi xy + x\ln z)\,[\pi x]$
$f_z = \sqrt{2}\cos(\pi xy + x\ln z)\left[\frac{x}{z}\right]$ so $\nabla f = \sqrt{2}\cos(\pi xy + x\ln z)\left(\pi y + \ln z,\ \pi x,\ \frac{x}{z}\right)$

For 4b)

$\mathbf{g}'(t) = (g_1'(t), g_2'(t), g_3'(t)) = \left(-\frac{1}{t^2}, \frac{1}{t^2}, \frac{1}{2}\right)$ for $t \neq 0$
OR

$$D\mathbf{g} = \begin{bmatrix} g_1' \\ g_2' \\ g_3' \end{bmatrix} = \begin{bmatrix} -\frac{1}{t^2} \\ \frac{1}{t^2} \\ \frac{1}{2} \end{bmatrix}, \text{ where } g_1(t) = \frac{1}{t}, g_2(t) = -\frac{1}{t}, \text{ and } g_3(t) = \frac{t}{2}$$

For 4c)

if $t = 2$ then $x = \frac{1}{2}, y = \frac{-1}{2}$ and $z = 1$ $\quad \sqrt{2}\cos(\pi xy + x\ln z) = \sqrt{2}\cos\frac{-\pi}{4} = 1$
and by Chain Rule

$$Dh = Df\ D\mathbf{g}$$

$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{2} & \frac{\pi}{2} & \frac{1}{2} \end{bmatrix}$ at $(\frac{1}{2}, -\frac{1}{2}, 1)$,and $\mathbf{g}'(2) = \left(\frac{-1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

$$h'(2) = Df(\tfrac{1}{2}, -\tfrac{1}{2}, 1)\mathbf{Dg}(2) = \begin{bmatrix} -\frac{\pi}{2} & \frac{\pi}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{-1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{4} = \frac{\pi + 1}{4}.$$

OR
$h'(2) = f_x x' + f_y y' + f_z z' = \nabla f(\frac{1}{2}, -\frac{1}{2}, 1) \bullet (g_1'(2), g_2'(2), g_3'(2)) =$
$= \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{1}{2}\right) \bullet \left(\frac{-1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{\pi + 1}{4}$

For 5)

Simplify the function first $\quad f(x, y) = \ln\frac{x^2 + y^2}{xy} = \ln\left(x^2 + y^2\right) - \ln x - \ln y$
for $x > 0, y > 0$ so
partials are $f_x = \frac{2x}{x^2 + y^2} - \frac{1}{x}$ $\quad f_y = \frac{2y}{x^2 + y^2} - \frac{1}{y}$
and $f_x(2, 1) = \frac{4}{5} - \frac{1}{2} = \frac{3}{10}$ $\quad f_y(2, 1) = \frac{2}{5} - 1 = \frac{-3}{5}$
Now, rate of change means the directional derivative
Since partials are continuous in the domain
$D_\nu f(2, 1) = \nabla f(2, 1) \bullet (\nu_1, \nu_2) = \left(\frac{3}{10}, -\frac{3}{5}\right) \bullet (\nu_1, \nu_2) = \frac{3}{10}(\nu_1 - 2\nu_2)$
and we are looking for a unit vector (ν_1, ν_2) such that
$D_\nu f(2, 1) = \frac{3}{10}$ $\quad \frac{3}{10}(\nu_1 - 2\nu_2) = \frac{3}{10}$ $\quad$ and $\nu_1^2 + \nu_2^2 = 1$
so $\quad \nu_1 - 2\nu_2 = 1$;one solution is $\nu_1 = 1, \nu_2 = 0$,
generally $\nu_1 = 1 + 2\nu_2 \Longrightarrow \nu_1^2 + \nu_2^2 = 5\nu_2^2 + 4\nu_2 + 1 = 1$,

$\nu_2(5\nu_2+4)=0$ thus another solution is $\nu_2=-\frac{4}{5}$ and $\nu_1=1-\frac{8}{5}=-\frac{3}{5}$
Together $\boldsymbol{\nu}=(1,0)$ or $\boldsymbol{\nu}=\frac{1}{5}(-3,-4)$

Maximum rate is always equal to $\|\nabla f\|=\left\|\left(\frac{3}{10},-\frac{3}{5}\right)\right\|=\frac{3}{10}\sqrt{1+4}=\frac{3\sqrt{5}}{10}.$

For 6)

$\nabla f=(f_x,f_y,f_z)=(2xze^y+z^2,\ x^2ze^y,\ x^2e^y+2xz)$ at P
$\nabla f(P)=(12,4,6)=2(6,2,3,)\qquad \|\nabla f\|=2\sqrt{49}=14$

the direction is $\mathbf{u}=-\dfrac{\nabla f}{\|\nabla f\|}=-\dfrac{1}{7}(6,2,3)$ and rate is $\|\nabla f\|=-14.$

For 7a)

$\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{2x^2+y^2}$ does not exists,for sure it is NOT equal to 0

since for $x\neq 0\qquad f(x,x)=\dfrac{1}{3}$ so f is **discontinous** at $(0,0)$;

For 7b)

by definition $\qquad f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-0}{x}=0$

$f_y(0,0)=\lim\limits_{y\to 0}\dfrac{f(0,y)-0}{y}=0$ so the gradient exists and $\nabla f\ (0,0)=(0,0)$.

For 7c)

for the directional derivative we have to use the definition
since f is discont.at $(0,0)$ thus partials cannot be cont.at$(0,0)$

unit vector in the direction of the line $y=x\qquad x=t,y=t\qquad$ is $\mathbf{u}=\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$

$D_uf(0,0)=\frac{1}{\sqrt{2}}\lim\limits_{t\to 0}\dfrac{f(t,t)-0}{t}=\dfrac{1}{2\sqrt{\ }}\lim\limits_{t\to 0}\dfrac{1}{t}\cdot\dfrac{t^2}{3t^2}=\lim\limits_{t\to 0}\dfrac{1}{3t}$does not exists

OR from a)

since $f(x,x)=\dfrac{1}{3}$ for $x\neq 0$ and $f(0,0)=0$ f is not continous along that line
so it cannot be differentiable along that line.

For 7d)

for any other point we can use the theorem since partials are continuous
$D_{\mathbf{u}}f=\nabla f\bullet\mathbf{u}$

first $\qquad f_x=y\dfrac{2x^2+y^2-4x^2}{(2x^2+y^2)^2}=y\dfrac{y^2-2x^2}{(2x^2+y^2)^2}$

and $\qquad f_y=x\dfrac{2x^2+y^2-2y^2}{(2x^2+y^2)^2}=x\dfrac{2x^2-y^2}{(2x^2+y^2)^2}$ so at the given point

$\nabla f(-1,-1)=\left(\dfrac{1}{9},-\dfrac{1}{9}\right)$ and $D_uf(-1,-1)=\dfrac{1}{9\sqrt{2}}(1,-1)\bullet(1,1)=0$