FINAL HANDOUT
MATH 349 SOLUTION .

1. For $a_n = \dfrac{n^3}{3^n}$ $\quad a_n > 0$,so not alternating;define $f(x) = \dfrac{x^3}{3^x}$ for $x > 0$

   then $\lim\limits_{x\to\infty} \dfrac{x^3}{3^x} =$ "$\frac{\infty}{\infty}$"(L'H.R.-3 times)$= \lim\limits_{x\to\infty} \dfrac{3x^2}{3^x \ln 3} = ..\ \lim\limits_{x\to\infty} \dfrac{6}{3^x (\ln 3)^3} =$ "$\frac{6}{\infty}$" $= 0$

   the sequence is convergent therefore bounded

   $f'(x) = \left(\dfrac{x^3}{3^x}\right)' = \dfrac{3x^2 3^x - x^3 3^x \ln 3}{(3^x)^2} = \dfrac{x^2(3 - x\ln 3)}{3^x} < 0$ if $x > \frac{3}{\ln 3} = 2.73$

   thus the sequence is ult.decr. for $n \geq 3$

   $a_1 = \frac{1}{3} < a_2 = \frac{8}{9} < a_3 = 1$ so an upper bound is 1, a lower bound is 0.

2. for $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-1)^n}{3^{3n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n(x-\frac{1}{3})^n}{3^{3n}\sqrt{n}}$.

   the centre is $c = \frac{1}{3}$ and the coefficients $a_n = \dfrac{1}{3^{2n}\sqrt{n}}$

   the radius of convergence $R = \dfrac{1}{L} = 9$

   since $L = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{3^{2n}\sqrt{n}}{3^{2n+2}\sqrt{n+1}} = \dfrac{1}{9}\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = \dfrac{1}{9}$

   thus the series is absolutely convergent on $\left(\frac{1}{3} - 9, \frac{1}{3} + 9\right) = \left(\frac{-26}{3}, \frac{28}{3}\right)$

   ends: $x = \frac{1}{3} + 9$ gives $\displaystyle\sum_{n=1}^{\infty} \frac{\left(x-\frac{1}{3}\right)^n}{3^{2n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent p-series $p = \frac{1}{2} < 1$

   for $x = \frac{1}{3} - 9$ gives $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is cond.convergent since the sequence $\frac{1}{\sqrt{n}} \searrow 0$

   (positive,decr.since $\frac{1}{\text{incr.}}$, and limit is $\frac{1}{\infty} = 0$)
   together the interval of convergence is $\left[\frac{-26}{3}, \frac{28}{3}\right)$, outside the series is divergent.

3. we know that $\quad e^s = \displaystyle\sum_{n=0}^{\infty} \frac{s^n}{n!}$ for any s ,use $s = x+1$

   then $f(x) = xe^x = [(x+1) - 1]\, e^{x+1} e^{-1} = \dfrac{1}{e}(x+1)\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} - \frac{1}{e}\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} =$

   $= \dfrac{1}{e}\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{n!} - \frac{1}{e}\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{(x+1)^k}{(k-1)!} - \frac{1}{e}\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} =$

   $= -\dfrac{1}{e} + \dfrac{1}{e}\displaystyle\sum_{k=1}^{\infty} \left[\frac{1}{(k-1)!} - \frac{1}{k!}\right](x+1)^k = -\frac{1}{e} + \frac{1}{e}\sum_{k=1}^{\infty} \frac{k-1}{k!}(x+1)^k$ for any x.

   check $a_0 = f(-1) = -e^{-1}$.

4. For the curve $c = \{z = x^2 + y^2\} \cap \{6x - 2y - z = 1\}$

for a) $z = x^2 + y^2 = 6x - 2y - 1$ so $x^2 + y^2 - 6x + 2y + 1 = 0$,complete the squares

$(x-3)^2 + (y+1)^2 = 9$ and $\left(\frac{x-3}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$ so

$x = 3 + 3\cos t, y = -1 + 3\sin t, z = 6x - 2y - 1 = 19 + 18\cos t - 6\sin t\ ,\ t \in [0, 2\pi)$

for b)we have two options to find $\overrightarrow{d}$ in $(x, y, z) = (0, -1, 1) + t\,\overrightarrow{d}$

from part a:

$t = \pi$ for the point $P = \overrightarrow{r}(\pi)$ and $\overrightarrow{d} = \overrightarrow{r}'(\pi) = (0, -3, 6)$ or $(0, 1, -2)$

since $\overrightarrow{r}'(t) = (-3\sin t, 3\cos t, -18\sin t - 6\cos t)$

OR

$\overrightarrow{d} = \overrightarrow{n_1} \times \overrightarrow{n_2}$ where $\overrightarrow{n_1} = \nabla F = (2x, 2y, -1)$ then at P $\qquad \overrightarrow{n_1} = (0, -2, -1)$

where $F(x, y, z) = x^2 + y^2 - z = 0$ and $G(x, y, z) = 6x - 2y - z = 1$

and $\quad \overrightarrow{n_2} = \nabla G = (6, -2, -1) \qquad \overrightarrow{n_1} \times \overrightarrow{n_2} = (0, 6, -12) = 6\,(0, 1, -2)$

so $(x, y, z) = (0, -1, 1) + t\,(0, 1, -2)$.

for c) for $P \qquad t = \pi$ and for $R \qquad t = \dfrac{\pi}{2}$,also

$$\|\overrightarrow{r}'(t)\| = \sqrt{9 + 36\,(3\sin t + \cos t)^2} = \sqrt{45 + 36(8\sin^2 t + 6\sin t\cos t)}$$

so $$s = \int_{\frac{\pi}{2}}^{\pi} \sqrt{45 + 72(4\sin^2 t + 3\sin t\cos t)}dt.$$

5. the equation $z = f(1, -1) + \nabla f(1, -1) \bullet (x - 1, y + 1) = 1 - (x - 1)$ gives

$z = 1 - (x - 1) = 2 - x$

since for $f(x, y) = e^{yx^2 \ln x} \qquad f(1, -1) = 1,$

$f_x(x, y) = e^{yx^2 \ln x}\,(2xy\ln x + yx) \qquad f_x(1, -1) = -1$

$f_y(x, y) = e^{yx^2 \ln x} x^2 \ln x \qquad f_y(1, -1) = 0$

OR

$\overrightarrow{n} = (\nabla f(1, -1), -1) = (-1, 0 - 1)$ and the point is $(1, -1, f(1, -1)) = (1, -1, 1)$

so $-x - z = d$ and through $P\,(1, -1, 1) \qquad x + z = 2.$

6. **for a)** at the origin we have to use the definition:

$$f_x(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x\to 0} \frac{1 - 0}{x} = \lim_{x\to 0^\pm} \frac{1}{x} \qquad DNE\ (\pm\infty)$$

$$f_y(0, 0) = \lim_{y\to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y\to 0} \frac{0 - 0}{y} = 0$$

thus the gradient does not exist at (0,0).

for b) since $f(x, x) = \frac{1}{4}$ for any $x \neq 0$ $\lim_{(x,y)\to(0,0)} f(x, y) \neq 0 = f(0, 0)$

the function is discont. at $(0, 0)$.

7. .

for the domain $\dfrac{x}{y} > 0$ both positive OR both negative

so the domain consists of the first and third quadrants without the axes

for level curves for any c

$c = \ln \frac{x}{y}, e^c = \frac{x}{y}, y = e^{-c}x$ lines through the origin

without the origin and with positive slopes $m = e^{-c} = 1, e, \frac{1}{e}$

since we got a level curve for any c the range is $z \in (-\infty, \infty)$.

8. $\nabla f = \left(2xe^{-y}, -e^{-y}\left(x^2 + \cos z\right), -e^{-y}\sin z\right)$ all partials are cont.functions so

$D_v f = \nabla f \bullet \boldsymbol{v}$ where $\nabla f\left(A\right) = (4, -3, 0)$

$\boldsymbol{v}$ is the unit vector in the direction of $\overrightarrow{AB} = (-3, -1, -\pi)$

and $\left\|\overrightarrow{AB}\right\| = \sqrt{10 + \pi^2}$ and finally

$D_v f\left(A\right) = \frac{1}{\sqrt{10+\pi^2}}\left(4, -3, 0\right) \bullet \left(-3, -1, -\pi\right) = \dfrac{-9}{\sqrt{10 + \pi^2}}$.

9. Since $x = uv^2, y = \frac{u}{v}$. for $u = 2, v = -1$

$x = 2$ $y = -2$ so we need $f_x(2, -2) = 3, f_y(2, -2) = -2$

$\dfrac{\partial F}{\partial u}\left(u, v\right) = \nabla f\left(x, y\right) \bullet \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = \left(f_x, f_y\right) \bullet \left(v^2, \frac{1}{v}\right)$

and $\dfrac{\partial F}{\partial u}\left(2, -1\right) = \left(3, -2\right) \bullet \left(1, -1\right) = 5.$

10. **for a)**

for $T = 0$ $y^2 - 3x^2 = 0$ $y = \pm x\sqrt{3}$ two lines

for $T = 3$ $y^2 - 3x^2 = 3$ hyperbola with intercepts $y = \pm\sqrt{3}, x = 0$

for $T = 6$ $y^2 - 3x^2 = -6$ hyperbola with intercepts $x = \pm\sqrt{2}, y = 0$

for b)

the direction is $-\nabla T(-1, 2)$

so first $\nabla T = (-6x, 2y)$ then $\nabla T\left(-1, 2\right) = (6, 4) = 2\left(3, 2\right)$ so $\mathbf{u} = \frac{-1}{\sqrt{13}}\left(3, 2\right)$.

for c)

the direction must be perpendicular to $\nabla T\left(-1, 2\right) = 2\left(3, 2\right)$

so $\mathbf{u} = \frac{\pm 1}{\sqrt{13}}\left(2, -3\right)$

11. Define $F_1(x, y, z) = xy^2 - z + u^2, F_2(x, y, z) = x^3z + 2y - u$

$F_3(x, y, z) = xu + y - xyz$.

since **all partials are continuous function** the only condition is that

$$\left\|\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}\right\| = \det\begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} \neq 0 \text{ at the given point}$$

so $\det\begin{bmatrix} y^2 & 2xy & -1 \\ 3x^2z & 2 & x^3 \\ u-yz & 1-xz & -xy \end{bmatrix}$ at the given point $= \det\begin{bmatrix} 1 & 2 & -1 \\ -3 & 2 & 1 \\ 2 & 2 & -1 \end{bmatrix} =$

along the first column

$$= 1\cdot\det\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} - (-3)\cdot\det\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + 2\cdot\det\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = -4+0+8 = 4 \neq 0.$$

12. For $f(x,y) = \ln(y\cos x)$ $x=0, y=1$

$T_2(x,y) = f(0,1) + \nabla f(0,1) \bullet (x, y-1) + \frac{1}{2}\left[Ax^2 + 2Bx(y-1) + C(y-1)^2\right]$

where $f(0,1) = \ln 1 = 0$ since $f(x,y) = \ln y + \ln\cos x$ for $y>0, \cos x > 0$

$f_x = \frac{-\sin x}{\cos x} = -\tan x$ $f_y = \frac{1}{y}$ $\nabla f(0,1) = (0,1)$

$f_{xx} = -\sec^2 x = \dfrac{-1}{\cos^2 x}$ $f_{xy} = 0$ $f_{yy} = -\dfrac{1}{y^2}$

and $A = f_{xx}(0,1) = -1, B = f_{xy} = 0$ $C = f_{yy}(0,1) = -1$

thus

$T_2(x,y) = 0 + (0,1) \bullet (x, y-1) + \frac{1}{2}\left[-x^2 + 0x(y-1) - (y-1)^2\right] =$

$$= y - 1 - \frac{x^2}{2} - \frac{(y-1)^2}{2}.$$

13. Define $F(x,y,z) = \arcsin(zy) + z^3x + x^2y + 8$ for $-1 \le zy \le 1, x$ any

then $\dfrac{\partial F}{\partial x} = z^3 + 2xy$ $\dfrac{\partial F}{\partial y} = \dfrac{z}{\sqrt{1-z^2y^2}} + x^2$ and $\dfrac{\partial F}{\partial z} = \dfrac{y}{\sqrt{1-z^2y^2}} + 3z^2x$

for $-1 < zy < 1, x$ any

all functions are cont. at the point $P(-1,0,2)$ and $\dfrac{\partial F}{\partial z}(P) = -12 \neq 0$ so the equation can be solved for z as a function of x, y around he point $P(-1,0,2)$

then $\dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\dfrac{2xy + z^3}{\frac{y}{\sqrt{1-z^2y^2}} + 3z^2x} = \dfrac{-8}{-12} = \dfrac{2}{3}$ at P and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{\frac{z}{\sqrt{1-z^2y^2}} + x^2}{\frac{y}{\sqrt{1-z^2y^2}} + 3z^2x} = -\frac{3}{-12} = \frac{1}{4} \text{ at that point..}$$