MATH 349
Handout # 2-Solutions.

1. the series $\sum\limits_{n=1}^{\infty} \dfrac{3^n \ln n}{n^n}$

   By Ratio test: $0 < \dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}\ln(n+1)}{(n+1)^{n+1}} \cdot \dfrac{n^n}{3^n \ln n} = \dfrac{\ln(n+1)}{\ln n} \cdot \dfrac{3}{n+1} \cdot \left(\dfrac{n}{n+1}\right)^n,$

   $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1 \cdot 0 \cdot \dfrac{1}{e} = 0 < 1$, the series is convergent

   (using $\lim\limits_{x\to\infty} \frac{\ln(x+1)}{\ln x} = 1$ by L'H Rule,and $\lim\limits_{n\to\infty}(1 - \frac{1}{n+1})^n = e^{-1}$).

2. In a) $\sum\limits_{n=2}^{\infty} \dfrac{1}{n(\ln n)^2}$;since $a_n = \dfrac{1}{n(\ln n)^2} > 0$ for $n \geq 2$ we can use Integral Test.

   The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is continous ,positive and decreasing for any $x \geq 2$

   because it is reciprocal of a positive,continous and increasing function(product of 2 incr.pos. funct-s)

   Now,$\displaystyle\int \frac{1}{x(\ln x)^2}dx = -\frac{1}{\ln x}$,using substitution $u = \ln x, du = \frac{1}{x}dx,$

   then $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}dx = -\lim_{x\to\infty}\frac{1}{\ln x} + \frac{1}{\ln 2} = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$.Therefore the series is convergent.

   In b) $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^2}$ In this case the integral is difficult ; try Comparison test:

   $\dfrac{1}{2}\ln n = \ln\sqrt{n} < \sqrt{n}$ for any $n \geq 2$,

   so $\ln n < 2\sqrt{n}$ and $0 < (\ln n)^2 < 4n$, $\dfrac{1}{(\ln n)^2} > \dfrac{1}{4n}$ and harmonic series is divergent

   so the given series is divergent.

3. $\sum\limits_{n=1}^{\infty} \frac{1}{n}\sin\frac{1}{n}$

   For $n \geq 1$, $\frac{1}{n} \leq 1$ and $0 < \sin\frac{1}{n} < \frac{1}{n}$ since $\sin x < x$ for $x > 0$.

   By Comparison Test $0 < \frac{1}{n}\sin\frac{1}{n} < \frac{1}{n^2}$ and $p = 2$ so $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is convergent and

   $\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\frac{1}{n}$ is also convergent.

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{n!n^n}$ is divergent by Ratio Test :

   $$0 < \frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!\,(n+1)^{n+1}} \cdot \frac{n!n^n}{(2n)!} = \frac{(2n+2)\,(2n+1)\,(2n)!}{(n+1)\,n!\,(n+1)\,(n+1)^n} \cdot \frac{n!n^n}{(2n)!} =$$

   $$= \frac{2\,(n+1)\,(2n+1)}{(n+1)\,(n+1)} \cdot \left(\frac{n}{n+1}\right)^n = \frac{2\,(2n+1)}{n+1}\left(1 - \frac{1}{n+1}\right)^n \to 2 \cdot 2 \cdot e^{-1} = \frac{4}{e} > 1$$

5. $\sum\limits_{n=1}^{\infty} \dfrac{e^n \cos^2 n}{\pi^n - 1}$

Since $0 < \cos^2 n < 1$,$0 < \dfrac{e^n \cos^2 n}{\pi^n - 1} \leq \dfrac{e^n}{\pi^n - 1} = a_n$,but this sequence is equivalent to the geometric sequence $b_n = \dfrac{e^n}{\pi^n}$,where $r = \dfrac{e}{\pi} < 1$

since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{e^n}{\pi^n - 1} \cdot \dfrac{\pi^n}{e^n} = \lim\limits_{n\to\infty} \dfrac{1}{1 - \frac{1}{\pi^n}} = 1 \neq 0$

By Limit Comparison Test $\sum\limits_{n=1}^{\infty} a_n$ is convergent

and by Comparison Test the original series is convergent.

6. Find the sum of $\sum\limits_{n=2}^{\infty} \dfrac{1}{e^{\frac{n}{2}}}$

the series is a geometric one where $r = \frac{1}{\sqrt{e}} < 1$ so the series is convergent and

$$\sum_{n=2}^{\infty} (\frac{1}{e^{\frac{1}{2}}})^n = \frac{(\frac{1}{\sqrt{e}})^2}{1 - \frac{1}{\sqrt{e}}} = \frac{1}{e} \cdot \frac{\sqrt{e}}{\sqrt{e} - 1} = \frac{1}{\sqrt{e}\,(\sqrt{e} - 1)}$$

using $\sum\limits_{n=N}^{\infty} r^n = \dfrac{r^N}{1 - r}$ for any $-1 < r < 1$.

7. the series $\sum\limits_{n=1}^{\infty} \dfrac{2 + \cos n}{\sqrt{n} + n}$ has positive terms and $1 \leq 2 + \cos n \leq 3$

Also $\quad 2 \cdot \sqrt{n} \leq \sqrt{n} + n \leq 2n$

so $\dfrac{1}{2n} \leq \dfrac{2 + \cos n}{\sqrt{n} + n} \leq \dfrac{3}{2\sqrt{n}}$ and we can use the left part of the inequality and Comparison test.Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n}$ is divergent

(a half of the harmonic series) and our series is bigger thus also divergent.

8. the series $\sum\limits_{n=1}^{\infty} \dfrac{5^n}{n^{n+1}}$ has positive terms so we can try Ratio or Root test

$\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+1}}{(n+1)^{n+2}} \cdot \dfrac{(n)^{n+1}}{5^n} = \dfrac{5}{(n+1)} \cdot \left(\dfrac{n}{n+1}\right)^{n+1} = \dfrac{5}{n+1} \cdot \left(1 - \dfrac{1}{n+1}\right)^{n+1} \rightarrow 0 \cdot e^{-1} = 0$

as $n \rightarrow \infty$. Since the limit $\rho = 0 < 1$ the series is convergent.

Root test is easier

$(a_n)^{\frac{1}{n}} = (\dfrac{5^n}{n^{n+1}})^{\frac{1}{n}} = \dfrac{5}{n^{1+\frac{1}{n}}} \rightarrow \dfrac{5}{\infty} = 0$ as $n \rightarrow \infty$ since $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

Since the limit $\sigma = 0 < 1$ the series is convergent.

9. Find the sum of $\sum\limits_{n=1}^{\infty} \frac{5+2^n}{5^{n+2}}$.We can split the series into two convergent ones:

$\sum\limits_{n=1}^{\infty} \frac{5+2^n}{5^{n+2}} = \sum\limits_{n=1}^{\infty} \frac{5}{5^{n+2}} + \sum\limits_{n=1}^{\infty} \frac{2^n}{5^{n+2}} = \frac{1}{5} \sum\limits_{n=1}^{\infty} \frac{1}{5^n} + \frac{1}{5^2} \sum\limits_{n=1}^{\infty} \frac{2^n}{5^n}$ both are convergent geometric series with $r = \frac{1}{5}$ and $r = \frac{2}{5}$ so the original series is covergent and the sum

$s = \frac{1}{5} \cdot \frac{\frac{1}{5}}{1-\frac{1}{5}} + \frac{1}{25} \cdot \frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{1}{20} + \frac{2}{75} = \frac{23}{300}$

using $\sum\limits_{n=N}^{\infty} r^n = \frac{r^N}{1-r}$ for any $-1 < r < 1$.

10. the series $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$ has positive terms for $n \geq 2$

the function $f(x) = \frac{\ln x}{x}$ is positive and continuous on $[2, \infty)$, since

$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x > e$ since $\ln x > \ln e = 1$,

so the function is decreasing on $[3, \infty)$ and we can use Integral test :

$\sum\limits_{n=3}^{\infty} \frac{\ln n}{n}$ is divergent since the integral (by subst.$u = \ln x, du = \frac{dx}{x}$)

$\int\limits_{3}^{\infty} \frac{\ln x}{x} dx = \int\limits_{\ln 3}^{\infty} u du = \left[\frac{u^2}{2}\right]_{\ln 3}^{\infty} = \infty$.

11. the series $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ has positive terms so we can try Ratio test:

$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2 \cdot (n!)^2}{(2n+2)(2n+1)\,(2n)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{n^2+2n+1}{4n^2+6n+2} \to \frac{1}{4}$

as $n \to \infty$ (divide the top and bottom by n^2). Since the limit $\rho = \frac{1}{4} < 1$

the series is convergent.