MATH 349
Handout #4 Solution

For 1)
$f(x,y) = \sqrt{2x+y^2}$
for the domain $2x+y^2 \geq 0 \qquad y^2 \geq -2x \qquad$ any $x>0$ will do,
generally the region right of the parabola $y^2 = -2x$, $x \leq 0$
level curves: $c=0 \qquad 0 = 2x+y^2$.... parabola above;
$c<0$.... NO curves; $c>0$...parabolas shifted to the right
$y^2 = c^2 - 2x$ ($y = \pm\sqrt{c^2-2x} \qquad$ the range is $[0,+\infty)$
Partials:
$$f_x = \frac{2}{2\sqrt{2x+y^2}} = \frac{1}{\sqrt{2x+y^2}}; f_y = \frac{2y}{2\sqrt{2x+y^2}} = \frac{y}{\sqrt{2x+y^2}}$$
obviously $\qquad y\ f_x = f_y$ in the domain except on $y^2 = -2x$.

For 2)
$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+2y^2} \qquad \text{for } (x,y) \neq (0,0)$$
define $g(x,y) = \dfrac{x^3}{x^2+2y^2}$ then $g(0,y) = 0$ for any $y \neq 0$ and for $x \neq 0$
$g\left(x,0\right) = \frac{x^3}{x^2} = x \to 0$ as $x \to 0$; along any line $y = mx$ for $x \neq 0$
$$g(x,mx) = \frac{x^3}{x^2+2m^2x^2} = \frac{x}{1+2m^2} \to 0 \text{ as } x \to 0.$$
Thus limit could be 0. Now,we have to prove it
$$|g(x,y) - 0| = \left|\frac{x^3}{x^2+2y^2}\right| = |x|\,\frac{x^2}{x^2+2y^2} \leq |x|\,\frac{x^2+2y^2}{x^2+2y^2} = |x| \to 0$$
as $x \to 0$ therefore $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+2y^2} = 0$ by Squeeze Theorem.

For 3)
For $f(x,y) = \dfrac{e^y}{x}$ the domain is $x \neq 0$,so $D = \mathbf{R}^2 - \{y - axis\}$
level curves: $c = 0$,NO curve since $e^y > 0$ always
for $c \neq 0 \qquad cx = e^y, \ln cx = y$,for $cx > 0$,and $y = \ln|x| + \ln|c|$
so for $c = 1, y = \ln x, x > 0$; for $c = -1, y = \ln(-x), x < 0$,
and for $c = e, y = \ln x + 1, x > 0$
the range is $(-\infty,0) \cup (0,+\infty)$

For the derivative $f_x = -\dfrac{e^y}{x^2}$,and $(f_x)_y = -\dfrac{e^y}{x^2} = f_{xy}$
since all functions are continuous in $D \qquad f_{xy} = f_{yx}$.

For 4)
for $f(x,y) = \left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{1}{2}}$
using Chain Rule $f_x = -\frac{1}{2}\left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (2xy + x^3)$ and
$f_y = -\frac{1}{2}\left(yx^2 + y^2 + \frac{1}{4}x^4\right)^{-\frac{3}{2}} \cdot (x^2+2y)$,so $\qquad xf_y = f_x$.
For the domain: $yx^2 + y^2 + \frac{1}{4}x^4 > 0$, $\left(y + \frac{1}{2}x^2\right)^2 > 0$,
always except when $y = -\frac{1}{2}x^2$

Also we can simplified using $\sqrt{a^2} = |a|$ $\quad sgn(a) = \frac{|a|}{a}$ for $a \neq 0$

$f(x,y) = \left|y + \frac{1}{2}x^2\right|^{-1}$,so $f_x = \dfrac{-x}{(y+\frac{1}{2}x^2)^2} sgn\left(y + \dfrac{1}{2}x^2\right)$,

and $f_y = \dfrac{-1}{(y+\frac{1}{2}x^2)^2} sgn\left(y + \dfrac{1}{2}x^2\right)$,if $y + \frac{1}{2}x^2 \neq 0$.

For 5)

For $f(x,y) = \ln(x^2 + y^2 + x)$

for domain solve :$x^2 + y^2 + x > 0$ complete the square: $\left(x + \frac{1}{2}\right)^2 + y^2 > \frac{1}{4}$

so the domain is outside the circle with the centre at $(-\frac{1}{2}, 0)$ and $r = \frac{1}{2}$.

For the level curves: $e^c = x^2 + y^2 + x$ so as above :

$\left(x + \frac{1}{2}\right)^2 + y^2 = e^c + \frac{1}{4}$

so all are circles with the same centre $(-\frac{1}{2}, 0)$ and $r = \sqrt{e^c + \frac{1}{4}}$,

for any c thus the range is $(-\infty, \infty)$

particularly if $c = 0$ $\quad r = \frac{\sqrt{5}}{2}$; $c = \ln 2$ $\quad r = \frac{3}{2}$;

$c = -\ln 2 = \ln \frac{1}{2}$ $\quad r = \frac{\sqrt{3}}{2}$...

Notice that cross-section $x = 0$ $\quad z = 2\ln|y|$

or for $y = 0$ $\quad z = \ln(x^2 + x)$

For 6)

define $g(x,y) = \dfrac{xy + y}{(x+1)^2 + y^2} = \dfrac{(x+1)y}{(x+1)^2 + y^2}$ for $(x,y) \neq (-1,0)$

then $g(-1,y) = \frac{0}{y^2} = 0$ for $y \neq 0$ and $g(x,0) = 0$ for $x \neq -1$

but for the line $\quad y = x + 1 \quad$ through the point $(-1, 0)$

$g(x, x+1) = \dfrac{(x+1)^2}{2(x+1)^2} = \dfrac{1}{2}$ for $x \neq -1$ so the limit DNE.

For 7)

$f(x,y) = \arctan \dfrac{x}{y}$ for $y \neq 0$

$f_x = \dfrac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \dfrac{1}{y} = \dfrac{y}{x^2 + y^2}$ and $f_y = \dfrac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \dfrac{-x}{y^2} = \dfrac{-x}{x^2 + y^2}$

so $\quad y\ f_x - x\ f_y = \dfrac{y^2 + x^2}{x^2 + y^2} = 1$ for any x and $y \neq 0$.

For 8)

For $f(x,y) = \dfrac{2x}{x^2 + y}$ the domain is

$D = \{y \neq -x^2\}$ ...$xy-$plane except the parabola $y = -x^2$

level curves $c = 0 \Longrightarrow 2x = 0, x = 0$...$y-$axis except the origin

for $c = 1$: $\quad 1 = \dfrac{2x}{x^2 + y}, \quad x^2 + y = 2x, \quad y = -x^2 + 2x = -x(x-2)$

.a parabola open down,vertex $V(1,1)$,roots $0, 2$ without the origin

for $c = -2$: $\quad -2 = \dfrac{2x}{x^2 + y}, \quad -2x^2 - 2y = 2x, \quad y = -x(1+x)$

a parabola open down ,vertex at $V(-\frac{1}{2}, \frac{1}{4})$,roots $0, -1$ without the origin

general level curves are shifted parabolas open down without the origin, passing through the origin $\qquad$ now,partials

$f_x(x,y) = 2 \cdot \dfrac{x^2+y-2x^2}{\left(x^2+y\right)^2} = \dfrac{2\left(y-x^2\right)}{\left(x^2+y\right)^2}$ at $x=-1, y=1$ $\quad f_x(-1,1)=0$

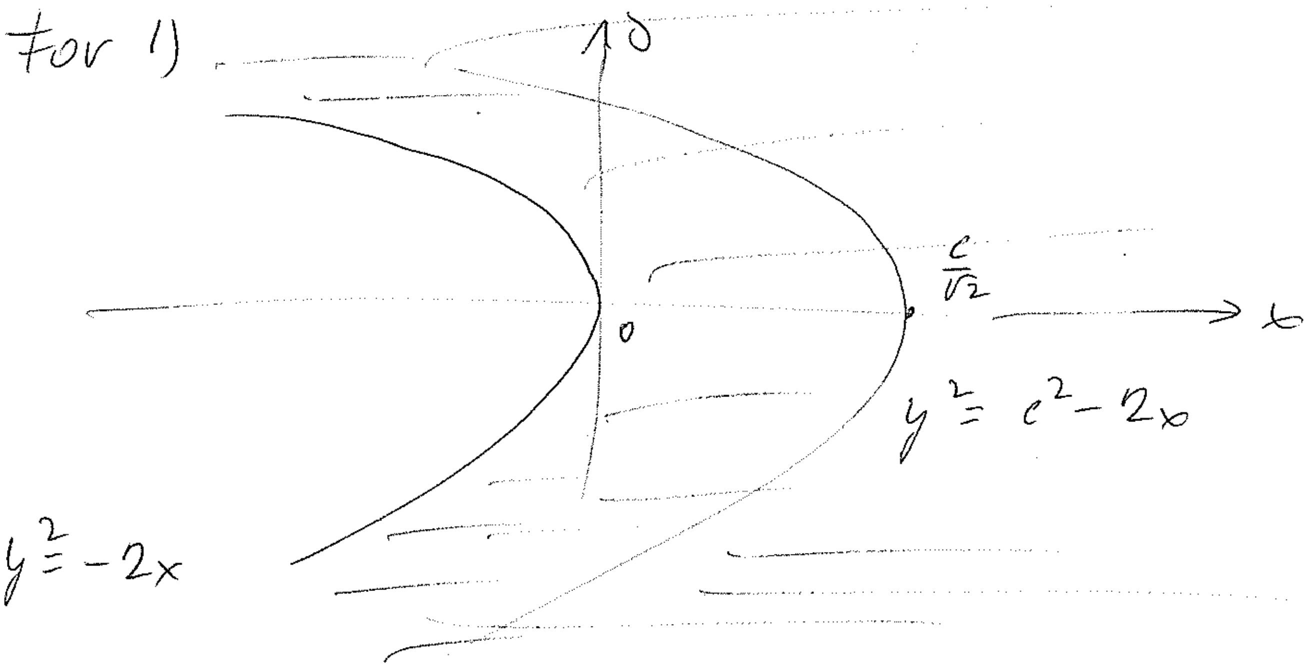
$f_y(x,y) = 2x \cdot (-1)\left(x^2+y\right)^{-2} \cdot 1 = \dfrac{-2x}{\left(x^2+y\right)^2}$ at $x=-1, y=1$ $\quad f_y(-1,1) = \frac{2}{4} = \frac{1}{2}$

so a normal vector to the tangent plane is $(f_x, f_y, -1) = \left(0, \frac{1}{2}, -1\right)$
or $\overrightarrow{n} = (0,1,-2)$ an equation is $\quad y-2z=d$
for d we need the point $z_0 = f(-1,1) = \frac{-2}{2} = -1 \qquad P(-1,1,-1)$
and $1+2=3=d$,so together $\quad y-2z=3.$

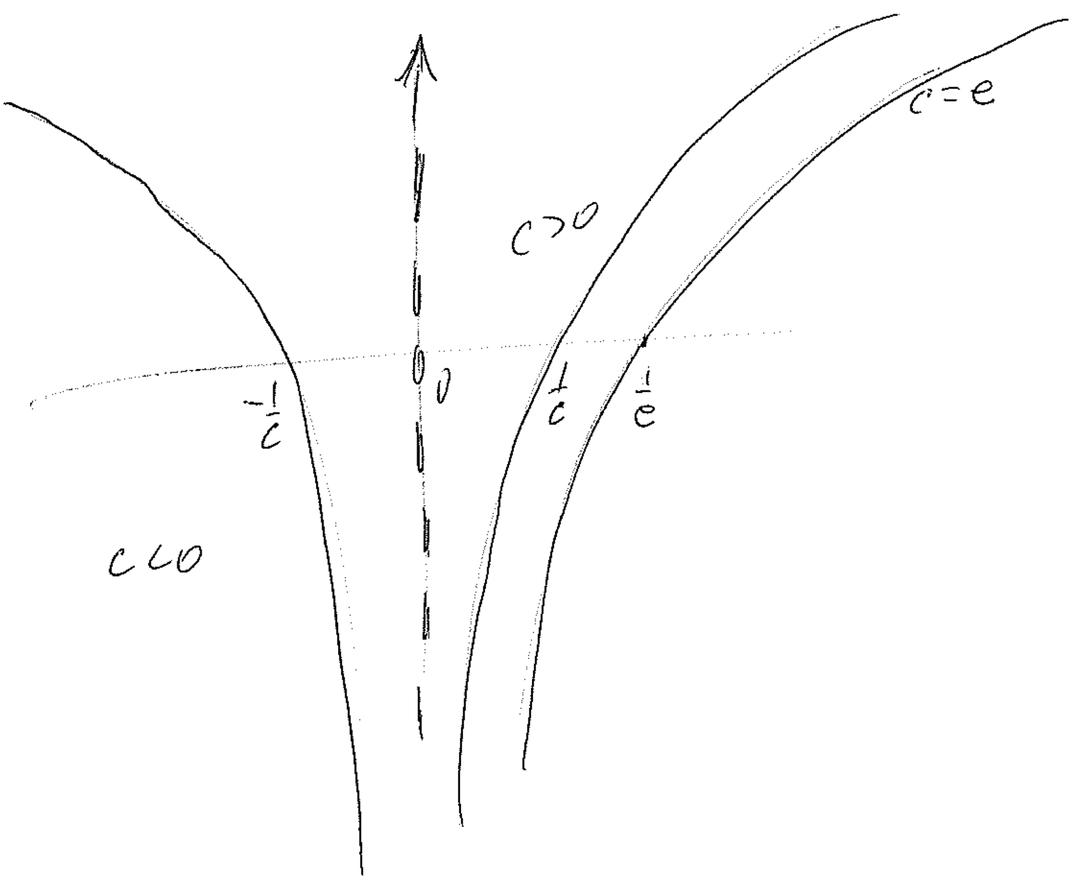
For 9)

define $g(x,y) = \dfrac{x(y-1)}{3x^2+2(y-1)^4}$ for $(x,y) \neq (0,1)$
then for $y \neq 1 \quad g(0,y)=0$ and for $x \neq 0 \quad g(x,1)=0,$
try a line through that point $\quad y-1=mx, m \neq 0$
for $x \neq 0 \quad f(x,mx+1) = \dfrac{mx^2}{3x^2+2m^4x^4} = \dfrac{m}{3+2m^4x^2} \to \dfrac{m}{3} \neq 0$
for any $m \neq 0 \qquad$ (as $x \to 0$)
Therefore the limit DNE.

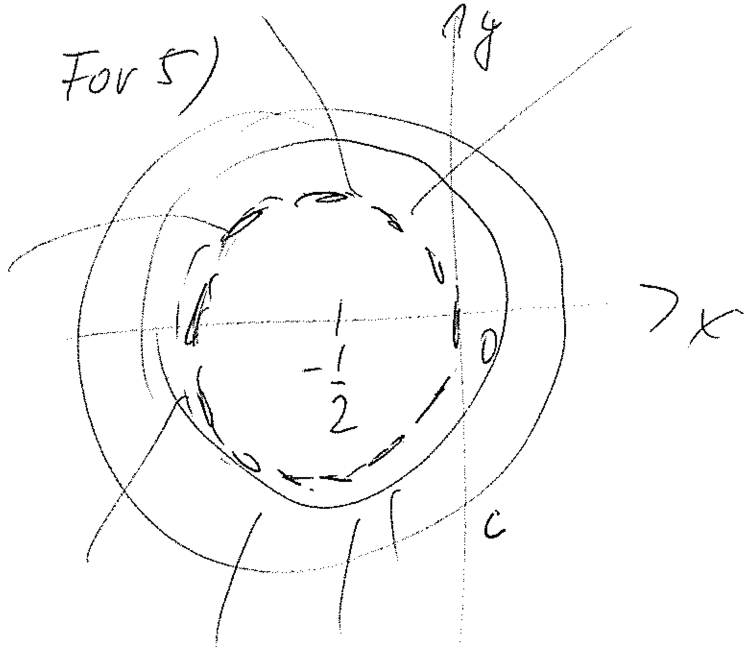
For 1)

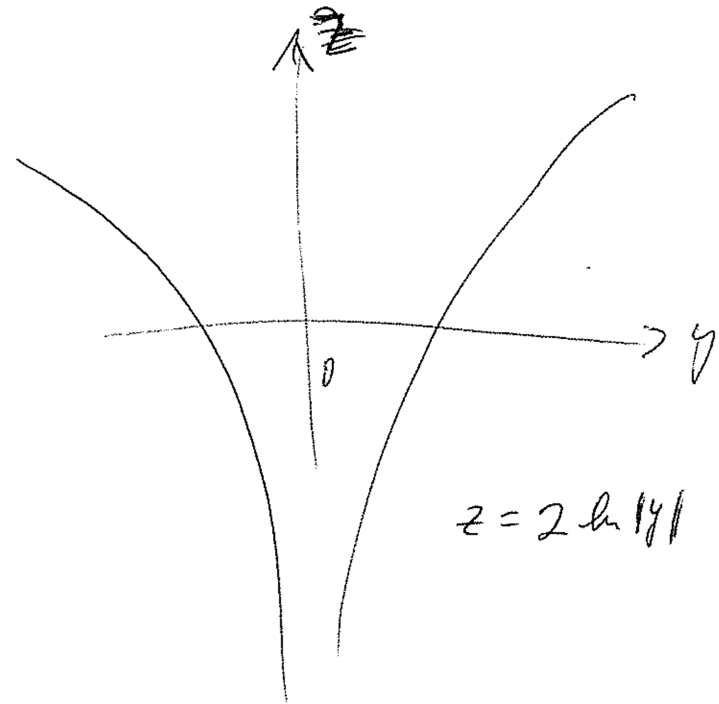


For 3)



For 5)





For 8)