MATH 349
Midterm Handout-Solution

1. **For the sequence** $a_n = \dfrac{\ell n(n+3)}{n+3}$

$\lim\limits_{n\to\infty} a_n = "\frac{\infty}{\infty}" L'H.R. = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x+3}}{1} = " \dfrac{1}{\infty} " = 0$

so the sequence is **convergent and thus bounded;**

For monotonicity ,define $f(x) = \dfrac{\ln x}{x}$

$f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} < 0$ for $\ln x > 1, x > e,$

thus the sequence is **decreasing** for $x = n+3 \geq 3$,$n \geq 1$

and an lower bound is 0,an upper bound is $a_1 = \frac{\ln 4}{4}$.

First abs.convergence $\sum\limits_{n=1}^{\infty}(\dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n+1}}) = 1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} + ... =$

$= 1 - \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1 - 0 = 1$(It is a telescoping series.)

$\sum\limits_{n=1}^{\infty} (-1)^n \left(\dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n+1}}\right)$ is **absolutely convergent** .

2. **For** $\sum\limits_{k=1}^{\infty} \dfrac{2^k}{\sqrt{k}} (x-1)^k, c = 1, a_k = \dfrac{2^k}{\sqrt{k}}$

$\left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{2^{k+1}}{\sqrt{k+1}} \cdot \dfrac{\sqrt{k}}{2^k} = \sqrt{\dfrac{k}{k+1}} \cdot 2 \to 2 \qquad R = \dfrac{1}{2}$

and the series is abs.convergent on $\left(\frac{1}{2}, \frac{3}{2}\right)$

For the ends $x = \frac{1}{2}$ or $\frac{3}{2}$ we get $\sum\limits_{k=1}^{\infty} (-1)^n \dfrac{1}{\sqrt{k}}$ or $\sum\limits_{k=1}^{\infty} \dfrac{1}{\sqrt{k}}$

the second one is divergent since $p = \frac{1}{2} < 1$,but the first one is cond.convergent since the sequence $\frac{1}{\sqrt{k}} \searrow 0$

Together the series is convergent on $\left[\frac{1}{2}, \frac{3}{2}\right)$

3. **For a) the series** $\sum\limits_{k=1}^{\infty} \dfrac{(\ell n\, 2)^k}{k!}$

we know that $e^s = \sum\limits_{\mathbf{k=0}}^{\infty} \dfrac{s^k}{k!}$ for any s so $\qquad e^s - 1 = \sum\limits_{\mathbf{k=1}}^{\infty} \dfrac{s^k}{k!}$

and for $s = \ln 2$ we get

$\sum\limits_{k=1}^{\infty} \dfrac{(\ell n\, 2)^k}{k!} = e^{\ln 2} - 1 = 2 - 1 = 1.$

For b) $\sum\limits_{n=3}^{\infty} \dfrac{(-1)^n}{2^n(n+1)}$ we need to know the sum of $\sum\limits_{\mathbf{n=3}}^{\infty} \dfrac{(x)^n}{(n+1)}$ for $x = -\frac{1}{2}$

and $\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = \frac{1}{x}\sum_{n=3}^{\infty} \frac{x^{n+1}}{(n+1)} = \frac{1}{x}\left[\sum_{\mathbf{n=0}}^{\infty} \frac{x^{n+1}}{(n+1)} - \frac{x^3}{3} - \frac{x^2}{2} - \frac{x}{1}\right]$

and since we know that $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)}$

$\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = -\frac{1}{x}\ln(1-x) - \frac{x^2}{3} - \frac{x}{2} - 1$ and finally for $x = -\frac{1}{2}$

$\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n(n+1)} = 2\ln\frac{3}{2} - \frac{1}{12} + \frac{1}{4} - 1 = 2\ln\frac{3}{2} - \frac{1-3+12}{12} = 2\ln\frac{3}{2} - \frac{5}{6}.$

USE Partial fraction first

$\frac{1}{(x+3)x} = \frac{1}{3}\left[\frac{-1}{x+3} + \frac{1}{x}\right] = \frac{1}{3}\left[\frac{-1}{(x+1)+2} + \frac{1}{(x+1)-1}\right] =$

$= \frac{1}{3}\left[\frac{-1}{2} \cdot \frac{1}{1+\frac{x+1}{2}} - \frac{1}{1-(x+1)}\right] =$

(using $\frac{1}{1+r} = \sum_{n=0}^{\infty}(-r)^n$ for $r = \frac{x+1}{2}$ and $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ for $r = x+1$)

$= -\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x+1}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}(x+1)^n = \frac{-1}{3}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} + 1\right](x+1)^n$

for $-1 < \frac{x+1}{2} < 1$ $\quad -2 < x+1 < 2$ and $-1 < x+1 < 1$,together

$-2 < x < 0$. And $a_6 = -\frac{1}{3}\cdot\left[\frac{1}{2^7} + 1\right] = -\frac{1+2^7}{3\cdot 2^7} = -\frac{129}{384}$.

By definition

$a_0 = f(2) = \ln\frac{1}{2} = -\ln 2 \qquad a_1 = f'(2) = \frac{1}{2}$

since $f'(x) = [\ln(x-1) - \ln x]' = \frac{1}{x-1} - \frac{1}{x}$, then $f''(x) = \frac{-1}{(x-1)^2} + \frac{1}{x^2}$

and $a_2 = \frac{1}{2}f''(2) = -\frac{3}{8}$, finally $f'''(x) = \frac{2}{(x-1)^3} - \frac{2}{x^3}$

and $a_3 = \frac{1}{6}f'''(2) = \frac{1}{3}\cdot\left(1 - \frac{1}{8}\right) = \frac{7}{24}$.

$T_3(x) = -\ln 2 + \frac{1}{2}(x-2) - \frac{3}{8}(x-2)^2 + \frac{7}{24}(x-2)^3$.

4. curve c given as the intersection of

   the cone $\{\ z = \sqrt{2x^2+2y^2}\}$ and the plane $\{z + x = 1\}$.

   from the plane $z = 1-x$ back to the cone $(1-x)^2 = 2x^2 + 2y^2$

   $1 = x^2 + 2x + 2y^2 \qquad 2 = (x+1)^2 + 2y^2 \qquad 1 = \left(\frac{x+1}{\sqrt{2}}\right)^2 + y^2$

   and then a parametrization is $\quad x = -1 + \sqrt{2}\cos t, y = \sin t$ and

   $z = 1 - x = 2 - \sqrt{2}\cos t \qquad t \in [0, 2\pi]$.

5. For the curve c given by $\mathbf{r}(t) = (2t, t^2, \ln t)\,, t > 0$

$t = 1$ for $P\,(2, 1, 0)$ $\qquad t = e$ for $R\,(2e, e^2, 1)$

find $\qquad \mathbf{r}'(t) = \left(2, 2t, \frac{1}{t}\right)$

then **for a)**

$\mathbf{d} = \mathbf{r}'(1) = (2, 2, 1)$ and the tangent line is

$(x, y, z) = (2, 1, 0) + s\,(2, 2, 1)$ or $x = 2 + 2z$ and $y = 1 + 2z$

for b) $\qquad$ for arclenght we need

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\left(\frac{2t^2 + 1}{t}\right)^2} = \frac{2t^2 + 1}{|t|}$$

and $\qquad s = \int\limits_1^e \frac{2t^2 + 1}{t} dt = \left[t^2 + \ln t\right]_1^e = e^2.$

6. For $\qquad \mathbf{r}\,(t) = (t \sin t, t \cos t, 2t)$

$\mathbf{r}'\,(t) = (\sin t + t \cos t, \cos t - t \sin t, 2)$ (product rule)

for the origin $t = 0$

for a) $\qquad$ so $\mathbf{d} = \mathbf{r}'(0) = (0, 1, 2)$

and an equation of the tangent is $\qquad (x, y, z) = t\,(0, 1, 2)$ or $x = 0, z = 2y$

For b)

for arclength we need $\|\mathbf{r}'\,(t)\|$

$\|\mathbf{r}'\,(t)\|^2 = (\sin t + t \cos t)^2 + (\cos t - t \sin t)^2 + 4 = \sin^2 t + 2t \sin t \cos t\; + t^2 \cos^2 t +$

$+ \cos^2 t - 2t \sin t \cos t\; + t^2 \sin^2 t\; + 4 = 5 + t^2$

now , $t = 0$ for the origin and $t = \frac{\pi}{2}$ for the point $A\left(\frac{\pi}{2}, 0, \pi\right)$

so arclength $\qquad s = \int\limits_0^{\frac{\pi}{2}} \|\mathbf{r}'\,(t)\|\, dt = \int\limits_0^{\frac{\pi}{2}} \sqrt{5 + t^2} dt = ($ Table $a = \sqrt{5})$

$$= \left[\tfrac{t}{2}\sqrt{5 + t^2} + \tfrac{5}{2} \ln\left(t + \sqrt{5 + t^2}\right)\right]_0^{\frac{\pi}{2}} = \tfrac{\pi}{4}\sqrt{5 + \tfrac{\pi^2}{4}} + \tfrac{5}{2} \ln\left(\tfrac{\pi}{2} + \sqrt{5 + \tfrac{\pi^2}{4}}\right) - \tfrac{5}{2} \ln \sqrt{5}.$$

7. Find a parametrization of the curve c given as the intersection of two surfaces

$c = \{x^2 + y^2 = 2z\,\} \cap \{3x - 4y - z = 0\}.$

from the plane $\qquad z = 3x - 4y \qquad$ into the paraboloid $x^2 + y^2 = 2(3x - 4y)$

$x^2 - 6x + y^2 + 8y = (x - 3)^2 + (y + 4)^2 - 25 = 0$

so $\qquad \left(\frac{x-3}{5}\right)^2 + \left(\frac{y+4}{5}\right)^2 = 1 \qquad$ thus a parametrization

$x = 3 + 5 \cos t \qquad y = -4 + 5 \sin t \qquad z = 25 + 15 \cos t - 20 \sin t, t \in [0, 2\pi)\,.$