DEPARTMENT OF MATHEMATICS AND STATISTICS

MIDTERM EXAM

MATH 349 LEC 01/02

Fall 2007 **TIME: 90 minutes**

Name:______________________________________ **I.D. No**.:____________
Total 80 **Each question is for 10 points.**

1. Is the sequence $a_n = \frac{1}{n}\cos(n\pi)$ convergent,ultimatelly monotonic,alternating,bounded?

   $cos(n\pi) = (-1)^n$ so the sequence is alternating thus NOT monotonic

   since $-\frac{1}{n} \leq \frac{1}{n}\cos(n\pi) \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{\pm 1}{n} = 0$ by Squ,Th. the sequence is also convergent to 0

   also a lower bound..... $-1 \leq -\frac{1}{n} \leq \frac{1}{n}\cos(n\pi) \leq \frac{1}{n} \leq 1$.....an upper bound

2. the series $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ by Integral test $\int_1^{\infty} \frac{1}{x}dx = [\ln x]_1^{\infty} = \infty$

   since $f(x) = \frac{1}{x}$ is decr. ,cont.,positve $=\frac{1}{incr.,pos.}$

   now, try Alt.Test $a_n = \frac{1}{n}$ is decr. sequence ,see above

   and $\lim_{n\to\infty} a_n = 0$ the series is **cond.convergent.**

3. $\sum_{n=0}^{\infty} \frac{\ln(2^n)}{5^n+1} = \sum_{n=0}^{\infty} \frac{n\ln 2}{5^n+1} = \ln 2 \sum_{n=0}^{\infty} \frac{n}{5^n+1}$

   the series has positive terms so try Ratio Test

   $\frac{a_{n+1}}{a_n} = \frac{n+1}{5^{n+1}+1} \cdot \frac{5^n+1}{n} = \frac{n+1}{n} \cdot \frac{5^n+1}{5^{n+1}+1} = \left(1+\frac{1}{n}\right)\frac{5^n+1}{5^{n+1}+1} \to \frac{1}{5}$

   SINCE

   $\lim_{n\to\infty} \frac{5^n+1}{5^{n+1}+1} \cdot \frac{\frac{1}{5^{n+1}}}{\frac{1}{5^{n+1}}} = \lim_{n\to\infty} \frac{\frac{1}{5}+\frac{1}{5^{n+1}}}{1+\frac{1}{5^{n+1}}} = \frac{1}{5}$ or by L'H.R. $\lim_{n\to\infty} \frac{5^n+1}{5^{n+1}+1} = \lim_{x\to\infty} \frac{5^x \ln 5}{5^{x+1}\ln 5} = \frac{1}{5}$

   and since the limit is less than 1 the series is convergent by Ratio test.

   ALSO

   $\ln(2^n) \leq 2^n$ $5^n + 1 > 5^n$ $\frac{\ln(2^n)}{5^n+1} < \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$ ..geom.series convergent

   since $\frac{2}{5} < 1$ and the original series is convergent by Comp.test

4. Find Taylor series around $c = -1$ for the function $f(x) = \dfrac{1}{(1-x)^3}$.Where is the expansion valid?

   we are looking for a_n such that $f(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$

   first $t = x + 1$ $x = t - 1$

   $$\frac{1}{1-x} = \frac{1}{2-t} = \frac{1}{2} \cdot \frac{1}{1-\frac{t}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(x+1)^n \text{ for } -1 < \tfrac{t}{2} < 1$$

   $-2 < x + 1 < 2$ thus $-3 < x < 1$

   now diffrentiate twice

   $$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}\frac{n}{2^{n+1}}(x+1)^{n-1} \qquad \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n+1}}(x+1)^{n-2}$$

   finally

   $$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n+2}}(x+1)^{n-2} = \sum_{k=0}^{\infty}\frac{(k+2)(k+1)}{2^{k+4}}(x+1)^k \qquad \text{for } -3 < x < 1.$$

5. Find the sum of $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n 2^{2n}}{n!} = \sum_{n=2}^{\infty}\frac{(-4)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-4)^n}{n!} - (-4) - (1) = e^{-4} + 3$

   using $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for any x.

6. For the curve c given by $\mathbf{r}(t) = (te^t, e^{-t}, e^{2t})$

   find an equation of the tangent line to c at the point $A(0,1,1)$.

   first $t = 0$ for A

   then $\mathbf{r}'(t) = (e^t + te^t, -e^{-t}, 2e^{2t})$ and $\mathbf{d} = \mathbf{r}'(0) = (1, -1, 2)$

   thus $(x, y, z) = (0, 1, 1) + s(1, -1, 2)$.

   Find a parametrization of the curve c given as the intersection of two surfaces $c = \{x^2 + y^2 = z\} \cap \{2x - y + z = 2\}$.

   from the plane $z = 2 - 2x + y$ into the paraboloid $x^2 + y^2 = 2 - 2x + y$

   $x^2 + 2x + y^2 - y = 2$ $(x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{13}{4}$

   so $x = -1 + \frac{\sqrt{13}}{2}\cos t$ $y = \frac{1}{2} + \frac{\sqrt{13}}{2}\sin t$ $z = \frac{9}{2} + \sqrt{13}\cos t - \frac{\sqrt{13}}{2}\sin t$

   for $t \in [0, 2\pi]$

7. Find the arclength of the curve c given by $\mathbf{r}(t) = (t^2, 2t^3, t^2)$ betweeen the origin and $P(1, -2, 1)$.

   first $t = 0$ for the origin $t = -1$ for the point P

   $\mathbf{r}'(t) = (2t, 6t^2, 2t)$

$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 36t^4 + 4t^2} = \sqrt{8t^2 + 36t^4} = \sqrt{4t^2(2 + 9t^2)} = 2|t|\sqrt{2 + 9t^2}$

$t \leq 0 \rightarrow |t| = -t$ finally, the arclength

$$s = \int_{-1}^{0} \|\mathbf{r}'(t)\| \, dt = -2\int_{-1}^{0} t\sqrt{2 + 9t^2}dt \qquad \text{subst.} u = 2 + 9t^2, du = 18tdt$$

thus $$s = -\tfrac{2}{18}\int_{11}^{2} \sqrt{u}du = \tfrac{1}{9}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_2^{11} = \tfrac{2}{27}\left[11^{\frac{3}{2}} - 2^{\frac{3}{2}}\right].$$