MATH 349
Handout # 1-solution

A
FOR 1)
For $a_n = \dfrac{e^n}{1+3^n}$ we can estimate: $0 < a_n < \dfrac{e^n}{3^n} = \left(\dfrac{e}{3}\right)^n$
by a geometric sequence with $r = \frac{e}{3} < 1$ so the limit is 0 and
by Squeeze theorem also $\lim a_n = 0$.
Back to the estimate :$0 < a_n < 1$ so a lower bound is 0 and an upper bound is 1
Also we can use L'Hopital Rule for the limit as $x \to \infty$:
$$\lim_{x\to\infty} \frac{e^x}{1+3^x} = \lim_{x\to\infty} \frac{e^x}{3^x \cdot \ln 3} = \frac{1}{\ln 3} \lim_{x\to\infty} e^{x(1-\ln 3)} = \text{"}e^{-\infty}\text{"} = 0 \text{ since } \ln 3 > 1.$$

FOR 2)
Let's call the sequence $x_n = \dfrac{2^n}{n!}$
then $0 < \dfrac{x_{n+1}}{x_n} = \dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n} = \dfrac{2}{n+1} < 1$ thus
$0 < x_{n+1} < x_n$ for all $n > 1$ so $0 < x_n < x_1 = 2$
therefore the sequence is bounded and decreasing.Thus the limit exists.
To find it estimate $x_n = \dfrac{2 \cdot 2 \cdot .... \cdot 2 \cdot 2}{n\,(n-1)\ldots \cdot 2 \cdot 1} \leq \dfrac{2}{n} \cdot 1.. \cdot 2 = \dfrac{4}{n}$ for $n > 2$
so by Squ.Th. the limit is 0.
FOR 3)
for a) $\{1, 2, 1, 3, 1, 4, 1.....1, n, 1...\}$or $\{n + (-1)^n\}_{n=1}^{\infty}$ or $\{(-1)^n n\}_{n=1}^{\infty}$
for b) $\{(-1)^n\}$ or $\{1, 2, 1, 2, 1....\}$
B
FOR 1)
For $a_n = \dfrac{n!}{n^n}$we can estimate:
$$0 < \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n < 1$$
thus $a_{n+1} < a_n$ so the sequence is decreasing
and $0 < a_n < a_1 = 1$ for all $n > 1$ so the sequence is bounded
for the limit estimate $0 < a_n = \dfrac{n!}{n^n} = \dfrac{n\,(n-1)\ldots 2 \cdot 1}{n \cdot n \cdot \ldots n \cdot n} \leq \dfrac{1}{n}$
so the limit is 0 by Squ.Th.
FOR 2)
For $a_n = \sqrt{n^2 - \frac{n}{3}} - n$ the type of the limit is "$\infty - \infty$" so rationalize
to change it to the type "$\frac{\infty}{\infty}$",
then divide both the top and bottom by the highest power in the denominator:
$$\left(\sqrt{n^2 - \tfrac{n}{3}} - n\right) \cdot \frac{\sqrt{n^2 - \frac{n}{3}} + n}{\sqrt{n^2 - \frac{n}{3}} + n} = \frac{n^2 - \frac{n}{3} - n^2}{\sqrt{n^2 - \frac{n}{3}} + n} = \frac{-\frac{n}{3}}{\sqrt{n^2 - \frac{n}{3}} + n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} =$$
$$\left(\text{using } \frac{1}{n} = \frac{1}{\sqrt{n^2}}\right) = \frac{-\frac{1}{3}}{\sqrt{1 - \frac{1}{3n}} + 1} \to \frac{-\frac{1}{3}}{2} = -\frac{1}{6}.$$
Since the sequence is convergent it must be bounded.From the last simplification

$a_n = \dfrac{-\frac{1}{3}}{\sqrt{1-\frac{1}{3n}}+1}$ so $a_n < 0$....upper bound, and $\sqrt{1-\frac{1}{3n}}+1 \geq 1$

(since $\sqrt{...}$ is always ≥ 0) $\qquad \dfrac{1}{\sqrt{1-\frac{1}{3n}}+1} \leq 1$

thus $a_n \geq -\dfrac{1}{3}$....lower bound. Together $-\frac{1}{3} \leq a_n < 0$.

FOR 3) for a) $\left\{-\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, ...\right\}$
for b) $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, -5, ...\}$

C

FOR 1).

For $a_n = \dfrac{n+(-1)^n}{n}$ $\qquad a_1 = 0, a_2 = \frac{3}{2}, a_3 = \frac{2}{3}, a_4 = \frac{5}{4}...$,

also $a_n = 1 + \frac{(-1)^n}{n}$, all terms positive except the first one,so not alternating,

but convergent since $\left|\frac{(-1)^n}{n}\right| \leq \frac{1}{n}$ and the limit of $\frac{1}{n}$ is 0.

So $\lim\limits_{n\to\infty} a_n = 1$. Thus the sequence must be bounded $0 \leq a_n \leq 1 + \frac{1}{n} \leq \frac{3}{2}$.

Now ,for even $n : a_n = 1 + \frac{(-1)^n}{n} = 1 + \frac{1}{n} > 1$

and for odd $n : a_n = 1 + \frac{(-1)^n}{n} = 1 - \frac{1}{n} < 1$ so the sequence is not monotonic.

FOR 2)

to find the limit of $a_n = \dfrac{4^n}{2^n+10}$ divide by 2^n the top and bottom

$$a_n = \frac{\frac{4^n}{2^n}}{1+\frac{10}{2^n}} = \frac{2^n}{1+\frac{10}{2^n}} \to \frac{\infty}{1+0} = +\infty$$

Also $\lim \dfrac{4^x}{2^x+10} = "\dfrac{\infty}{\infty}"(L'H) = \lim \dfrac{4^x \ln 4}{2^x \ln 2} = \lim 2^x \cdot 2 = +\infty$

as $x \to +\infty$

also the sequence is increasing since $a_n < a_{n+1}$

$$\frac{4^n}{2^n+10} < \frac{4^{n+1}}{2^{n+1}+10} \qquad 2^{n+1}+10 < 4\cdot 2^n + 40 \qquad 0 < 2^{n+1}+30$$

FOR 3).

Give an example of a sequence which is divergent and bounded.

$\{1, 2, 1, 2, ...\}$ or $\{(-1)^n\}$

or any mixture of two convergent therefore bounded sequences:

$a_n = \frac{1}{n}$ for n odd and $a_n = \frac{n}{n+1}$ for n even i.e.$\left\{1, \frac{2}{3}, \frac{1}{3}, \frac{4}{5}, \frac{1}{5}, \frac{6}{7}\right\}$

so odd terms have limit 0,but even terms have limit 1,

so the whole sequence has NO limit.

D

FOR 1)

$\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty}(x - 2^x) = "\infty - \infty"$

so we have to compare ,but we know that exp.function is stronger

than any polynomial so limit is $-\infty$, to prove it:

$\lim\limits_{x\to\infty} 2^x \cdot (\dfrac{x}{2^x} - 1) = \left(\lim\limits_{x\to\infty} 2^x\right)(L-1)$,

where $L = \lim_{x\to\infty} \frac{x}{2^x}$ and we can use L'Hopital Rule since the type is "$\frac{\infty}{\infty}$",so
$L = \lim_{x\to\infty} \frac{1}{2^x \cdot \ln 2} = 0$ since $\frac{1}{\infty} = 0$. Together $\lim_{n\to\infty} a_n = +\infty \cdot (-1) = -\infty$.
So the sequence is divergent and not bounded below.Is it bounded above?
Investigate: $a_1 = -1, a_2 = -3$, all terms are negative so an upper bound is 0,
to see it compare the graphs $y = x$ and $y = 2^x$, the line is always below exp.function
Also $a_n \le a_1 = -1$ since the sequence is decreasing: $a_{n+1} < a_n$ proof:
$n + 1 - 2^{n+1} < n - 2^n \qquad 1 < 2^n(2 - 1) \qquad 1 < 2 \le 2^n$ for all n.

FOR 2). For $b_n = (n+1)^{\frac{1}{n}} \qquad b_n = f(n)$, where $f(x) = e^{\frac{1}{x}\ln(x+1)}$.
Calculate the limit of the exponent first:

$$L = \lim_{x\to\infty} \frac{\ln(x+1)}{x} = \text{"}\frac{\infty}{\infty}\text{"} L'H = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{1} = 0, \text{ so } \lim_{n\to\infty} b_n = e^0 = 1$$

for monotonicity: $$f'(x) = e^{\frac{1}{x}\ln(x+1)} \left[\frac{\frac{x}{x+1} - \ln(x+1)}{x^2} \right] < 0$$

since $$\frac{x}{x+1} - \ln(x+1) = \frac{x+1-1}{x+1} - \ln(x+1) = 1 - \frac{1}{x+1} - \ln(x+1) < 0$$
if $1 < \ln(x+1)$ which is true for sure for $x \ge 2$
thus the sequence is decreasing;
also by definition $b_{n+1} < b_n \qquad (n+2)^{\frac{1}{n+1}} < (n+1)^{\frac{1}{n}}$
$(n+2)^n < (n+1)^{n+1} \qquad \left(\frac{n+2}{n+1}\right)^n < n+1$
$\left(1 + \frac{1}{n+1}\right)^n < n+1$ using the binomial formula

$$\sum_{k=0}^{k=n} \binom{n}{k} \left(\frac{1}{n+1}\right)^k = \sum_{k=0}^{k=n} \frac{1}{k!} \cdot \frac{n\,(n-1)\,..\,(n-k+1)}{(n+1)\,(n+1)\,.....(n+1)} < \sum_{k=0}^{k=n} 1 = n+1$$

FOR 3)
For a) the sequence must be convergent e.g. $a_n = 1 - \frac{1}{n}$
and it is increasing since $\left\{\frac{1}{n}\right\}$ is decreasing
For b) $a_n = 3 + (-1)^n \frac{1}{n}$ since $\frac{1}{n} \to 0$, and also
the alternating sequence $\frac{(-1)^n}{n} \to 0$ by Squeeze Theorem