The University of Calgary
Department of Mathematics and Statistics
MATH 349
Handout # 3 Solution

For 1a)

$\sum_{n=3}^{\infty} |...| = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)}$ is divergent by Integral test:

$f(x) = \frac{1}{x \ln x}$ is positive,continous and decreasing for $x > 1$

since $x \ln x$ is the product of two positive and increasing functions,

The integral is divergent,since

$$\int_3^{\infty} \frac{1}{x(\ln x)} dx = (u = \ln x, du = \frac{dx}{x}, \int \frac{du}{u} = \ln |u|) = \lim_{x\to\infty} \ln(\ln x) - \ln\ln 3 = \infty$$

so the original series is NOT absolutely convergent.

For 1b)

it is conditionally convergent by Alt.Test since $a_n = \frac{1}{n(\ln n)} \to 0 \left(\frac{1}{\infty}\right)$

and the sequence is decreasing — see above f is decreasing function

For 2a)

the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ is divergent by Comparison Test since $0 < \ln x < x$ for $x > 1$,

$\ln(n+1) < n+1 \qquad \frac{1}{\ln(n+1)} > \frac{1}{n+1}$

and $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{k=2}^{\infty} \frac{1}{k}$ which is divergent harmonic series (by Integral Test).

For 2 b)

the series is conditionally convergent by Alternating Test since the sequence

$a_n = \frac{1}{\ln(n+1)}$ has positive terms,limit 0 $\left(\frac{1}{\infty}\right)$and is decreasing

since $\ln x$ is increasing,positive for $x > 1$, and "$\frac{1}{\text{incr,pos}}$" = decr.

For 3a)

investigate $\sum_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n!})$ since $a_n = \left(\frac{1}{n} - \frac{1}{n!}\right) > 0$ for $n \geq 3$

We can split into two series

harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ which is divergent and the series $\sum_{n=2}^{\infty} \frac{1}{n!}$ which is convergent

by Ratio Test: $0 < \frac{1}{(n+1)!} \cdot n! = \frac{1}{n+1} \to 0 < 1$,

so together the series $\sum_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n!})$ is divergent.

The original series is NOT abs. convergent.

Also by Comparison Test for $n \geq 3$ $n! \geq 2n$ so $\frac{1}{n!} \leq \frac{1}{2n}$ and

$\frac{1}{n} - \frac{1}{n!} \geq \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \qquad a_n \geq \frac{1}{2n}$ for $n > 2$.

For 3b)

we can separate again since $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n}$ is conditionally convergent by Alternating Test:

the sequence $\left\{\frac{1}{n}\right\}$ is decreasing,with positive terms and limit 0.
And the second series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n!}$ is absolutely convergent from above ,
so together the original series is **conditionally convergent**.
Also directly by Alternating test but it is harder
the sequence a_n from above has limit 0 ,so we have to show that it is decreasing:
$a_{n+1} < a_n \qquad \frac{1}{n+1} - \frac{1}{(n+1)!} < \frac{1}{n} - \frac{1}{n!}$, multiply both sides by $(n+1)!$:
$n! - 1 < (n+1)(n-1)! - (n+1)$, so $n! < n(n-1)! + (n-1)! - n$
$0 < (n-1)! - n$
finally $n < (n-1)(n-2) < (n-1)!$ which is true for $n \geq 4$

For 4a)

the centre is $c = -1$ and $a_n = \dfrac{n!}{4^n}$, so $0 < \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{4^{n+1}} \cdot \dfrac{4^n}{n!} = \dfrac{n+1}{4} \to \infty$,
so $R = \frac{1}{L} = 0$ and the series converges ONLY for $x = -1$.

For 4b)

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}(2x-1)^n = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} 2^n (x - \frac{1}{2})^n$$

the centre is $c = \frac{1}{2}$ and $a_n = \dfrac{n! \cdot 2^n}{(2n)!}$, so $0 < \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!2^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{n!2^n} =$
$= \dfrac{(n+1)2}{(2n+2)(2n+1)} = \dfrac{1}{2n+1} \to 0$ so $R = \frac{1}{L} = +\infty$, so the interval is $(-\infty, +\infty)$.
and the series is abs.convergent for any x.

For 5)

The answer must be in the form $\sum_{n=1}^{\infty} a_n (x-1)^n$
first

$$x - 1 = t \qquad x = t + 1 \qquad \frac{1}{(x+1)^2} = \frac{1}{(t+2)^2} = \frac{1}{4} \cdot \frac{1}{(1+\frac{t}{2})^2}$$

from Table

$$\sum_{n=1}^{\infty} (-1)^{n-1} n r^{n-1} = \frac{1}{(1+r)^2} \text{ for any } -1 < r < 1 \text{ and } r = \frac{t}{2}$$

$$\frac{1}{(x+1)^2} = \frac{1}{4} \sum_{\substack{n=1\\1}}^{\infty} (-1)^{n-1} n \frac{(x-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}} \cdot n (x-1)^{n-1}$$

$OR \qquad (n-1 = k)$

$$\frac{1}{(x+1)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{2^{k+2}} (x-1)^k \qquad \text{for } -1 < x < 3$$

since $-1 < \dfrac{t}{2} < 1 \qquad -2 < x - 1 < 2$.

For 6a)

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n (x - \frac{1}{4})^n}{n^n} \qquad \text{the centre is } c = \tfrac{1}{4} \text{ and } a_n = \frac{4^n}{n^n}.$$

Since $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{4^{n+1}}{(n+1)^{n+1}} \cdot \dfrac{n^n}{4^n} = \dfrac{4}{n+1} \cdot \left(\dfrac{n}{n+1}\right)^n \to L = 0 \cdot \dfrac{1}{e} = 0$,
using $\lim_{n\to\infty} \left(1 + \dfrac{a}{n}\right)^n = e^a \qquad R = \frac{1}{L} = +\infty$, the interval is $(-\infty, +\infty)$

the series is abs. convergent for any x;
Easier by Root Test $(|a_n|)^{\frac{1}{n}} = \frac{4}{n} \to 0$.

For 6b)

the centre is $c = 4$ and $a_n = (-1)^n \frac{n}{2^n}$,since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \to \frac{1}{2}$
the radius is $R = 2$ and series is absolutely convergent on $(2, 6)$.
Now,for $x = 2$
we have to investigate $\sum_{n=1}^{\infty} n = +\infty$ and for $x = 6$ the series $\sum_{n=1}^{\infty} (-1)^n n$ which is also divergent since ther limit of the n-th term is NOT 0.
So the interval is only $(2, 6)$

For 7)

The answer must be in the form $\sum_{n=0}^{\infty} a_n (x+1)^n$.
put $x + 1 = t \qquad x = t - 1$ then
$\ln(2 - x) = \ln(3 - t) = \ln\left[3(1 - \frac{t}{3})\right] = \ln 3 + \ln(1 - \frac{t}{3})$
using $\ln(1 - s) = -\sum_{n=1}^{\infty} \frac{1}{n} s^n$ for $s \in [-1, 1) \qquad s = \frac{x+1}{3}$
$= \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n 3^n} (x+1)^n$ for $x \in [-4, 2[$
since $-1 \leq \frac{x+1}{3} < 1 \qquad -3 \leq x + 1 < 3 \qquad -4 \leq x < 2$

For 8)

The answer must be in the form $\sum_{n=0}^{\infty} a_n (x+4)^n$.
put $x + 4 = t \qquad x = t - 4$

$$\frac{1}{1-2x} = \frac{1}{1-2(t-4)} = \frac{1}{9-2t} = \frac{1}{9} \cdot \frac{1}{1 - \frac{2}{9}t}$$

now,using $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ for $-1 < r < 1$,where $r = \frac{2(x+4)}{9}$ we get

$$\frac{1}{1-2x} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{2^n}{9^n} (x+4)^n = \sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}} (x+4)^n, \text{ so } a_n = \frac{2^n}{9^{n+1}}$$

Now , to find the interval ,solve for x: $\qquad -1 < \frac{2(x+4)}{9} < 1$
$-9 < 2x + 8 < 9 \qquad \frac{-17}{2} < x < \frac{1}{2}$.