

MATH 349
Handout #4 Solution

For 1)

$$f(x, y) = \sqrt{2x + y^2}$$

for the domain $2x + y^2 \geq 0$ $y^2 \geq -2x$ any $x > 0$ will do,

generally the region right of the parabola $y^2 = -2x$, $x \leq 0$

level curves:

$c = 0$ $0 = 2x + y^2$ parabola above;

$c < 0$ NO curves;

$c > 0$...parabolas shifted to the right

$$y^2 = c^2 - 2x \quad (y = \pm\sqrt{c^2 - 2x} \quad \text{the range is } [0, +\infty))$$

Partials:

$$f_x = \frac{2}{2\sqrt{2x + y^2}} = \frac{1}{\sqrt{2x + y^2}}; f_y = \frac{2y}{2\sqrt{2x + y^2}} = \frac{y}{\sqrt{2x + y^2}}$$

obviously

$$y f_x = f_y \text{ in the domain except on } y^2 = -2x.$$

For 2)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + 2y^2} \quad \text{for } (x, y) \neq (0, 0)$$

define $g(x, y) = \frac{x^3}{x^2 + 2y^2}$ then $g(0, y) = 0$ for any $y \neq 0$ and for $x \neq 0$

$g(x, 0) = \frac{x^3}{x^2} = x \rightarrow 0$ as $x \rightarrow 0$; along any line $y = mx$ for $x \neq 0$

$$g(x, mx) = \frac{x^3}{x^2 + 2m^2x^2} = \frac{x}{1 + 2m^2} \rightarrow 0 \text{ as } x \rightarrow 0.$$

Thus limit could be 0. Now, we have to prove it

$$|g(x, y) - 0| = \left| \frac{x^3}{x^2 + 2y^2} \right| = |x| \frac{x^2}{x^2 + 2y^2} \leq |x| \frac{x^2 + 2y^2}{x^2 + 2y^2} = |x| \rightarrow 0$$

as $x \rightarrow 0$ therefore $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + 2y^2} = 0$ by Squeeze Theorem.

For 3)

For $f(x, y) = \frac{e^y}{x}$ the domain is $x \neq 0$, so $D = \mathbf{R}^2 - \{y\text{-axis}\}$

level curves:

$c = 0$ NO curve since $e^y > 0$ always

for $c \neq 0$ $cx = e^y$ $\ln cx = y$ for $cx > 0$,

also $y = \ln |x| + \ln |c|$

so for $c = 1, y = \ln x, x > 0$ for $c = -1, y = \ln(-x), x < 0$

and for $c = e, y = \ln x + 1, x > 0$ the range is $(-\infty, 0) \cup (0, +\infty)$

For the derivative

$$f_x = -\frac{e^y}{x^2} \quad (f_x)_y = -\frac{e^y}{x^2} = f_{xy}$$

since all functions are continuous in D $f_{xy} = f_{yx}$.

For 4)

$$\text{for } f(x, y) = (yx^2 + y^2 + \frac{1}{4}x^4)^{-\frac{1}{2}}$$

using Chain Rule $f_x = -\frac{1}{2}(yx^2 + y^2 + \frac{1}{4}x^4)^{-\frac{3}{2}} \cdot (2xy + x^3)$ and

$$f_y = -\frac{1}{2}(yx^2 + y^2 + \frac{1}{4}x^4)^{-\frac{3}{2}} \cdot (x^2 + 2y), \text{ so } xf_y = f_x.$$

For the domain:

$$yx^2 + y^2 + \frac{1}{4}x^4 > 0, \quad (y + \frac{1}{2}x^2)^2 > 0,$$

always except when $y = -\frac{1}{2}x^2$

Also we can simplified using $\sqrt{a^2} = |a|$ $\text{sgn}(a) = \frac{|a|}{a}$ for $a \neq 0$

$$f(x, y) = |y + \frac{1}{2}x^2|^{-1}, \text{ so } f_x = \frac{-x}{(y + \frac{1}{2}x^2)^2} \operatorname{sgn}(y + \frac{1}{2}x^2),$$

$$\text{and } f_y = \frac{-1}{(y + \frac{1}{2}x^2)^2} \operatorname{sgn}(y + \frac{1}{2}x^2), \text{ if } y + \frac{1}{2}x^2 \neq 0.$$

For 5)

$$\text{For } f(x, y) = \ln(x^2 + y^2 + x)$$

for domain solve $x^2 + y^2 + x > 0$ complete the square: $(x + \frac{1}{2})^2 + y^2 > \frac{1}{4}$
so the domain is outside the circle with the centre at $(-\frac{1}{2}, 0)$ and $r = \frac{1}{2}$.

For the level curves: $e^c = x^2 + y^2 + x$ so as above :

$$(x + \frac{1}{2})^2 + y^2 = e^c + \frac{1}{4}$$

so all are circles with the same centre $(-\frac{1}{2}, 0)$ and $r = \sqrt{e^c + \frac{1}{4}}$,

for any c thus the range is $(-\infty, \infty)$

$$\text{particularly if } c = 0 \quad r = \frac{\sqrt{5}}{2}; c = \ln 2 \quad r = \frac{3}{2};$$

$$c = -\ln 2 = \ln \frac{1}{2} \quad r = \frac{\sqrt{3}}{2} \dots$$

Notice that cross-section $x = 0 \quad z = 2 \ln |y|$

or for $y = 0 \quad z = \ln(x^2 + x)$

For 6)

$$\text{define } g(x, y) = \frac{xy + y}{(x + 1)^2 + y^2} = \frac{(x + 1)y}{(x + 1)^2 + y^2} \text{ for } (x, y) \neq (-1, 0)$$

then $g(-1, y) = \frac{0}{y^2} = 0$ for $y \neq 0$ and $g(x, 0) = 0$ for $x \neq -1$

but for the line

$$y = x + 1 \quad \text{through the point } (-1, 0)$$

$$g(x, x + 1) = \frac{(x + 1)^2}{2(x + 1)^2} = \frac{1}{2} \text{ for } x \neq -1 \text{ so the limit DNE.}$$

For 7)

$$f(x, y) = \arctan \frac{x}{y} \text{ for } y \neq 0$$

$$f_x = \frac{1}{1 + (\frac{x}{y})^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2} \text{ and } f_y = \frac{1}{1 + (\frac{x}{y})^2} \cdot \frac{-x}{y^2} = \frac{-x}{x^2 + y^2}$$

$$\text{so } y f_x - x f_y = \frac{y^2 + x^2}{x^2 + y^2} = 1 \text{ for any } x \text{ and } y \neq 0.$$

For 8)

$$\text{For } f(x, y) = \frac{2x}{x^2 + y} \text{ the domain is}$$

$$D = \{y \neq -x^2\} \dots xy\text{-plane except the parabola } y = -x^2$$

level curves $c = 0 \implies 2x = 0, x = 0 \dots y\text{-axis except the origin}$

$$\text{for } c = 1: \quad 1 = \frac{2x}{x^2 + y}, \quad x^2 + y = 2x, \quad y = -x^2 + 2x = -x(x - 2)$$

.a parabola open down, vertex $V(1, 1)$, roots 0, 2 without the origin

for $c = -2$:

$$-2 = \frac{2x}{x^2 + y}, \quad -2x^2 - 2y = 2x, \quad y = -x(1 + x)$$

a parabola open down, vertex at $V(-\frac{1}{2}, \frac{1}{4})$, roots 0, -1 without the origin

general level curves are shifted parabolas open down without the origin, passing through the origin

now, partials

$$f_x(x, y) = 2 \cdot \frac{x^2 + y - 2x^2}{(x^2 + y)^2} = \frac{2(y - x^2)}{(x^2 + y)^2} \text{ at } x = -1, y = 1 \quad f_x(-1, 1) = 0$$

(by Quotient Rule)

$$f_y(x, y) = 2x \cdot (-1) (x^2 + y)^{-2} \cdot 1 = \frac{-2x}{(x^2 + y)^2} \text{ at } x = -1, y = 1 \quad f_y(-1, 1) = \frac{2}{4} = \frac{1}{2}$$

(by Chain Rule)

so a normal vector to the tangent plane is $(f_x, f_y, -1) = (0, \frac{1}{2}, -1)$

or $\vec{n} = (0, 1, -2)$ an equation is $y - 2z = d$

for d we need the point $z_0 = f(-1, 1) = \frac{-2}{2} = -1$ $P(-1, 1, -1)$

and $1 + 2 = 3 = d$, so together $y - 2z = 3$.

For 9)

define $g(x, y) = \frac{x(y-1)}{3x^2 + 2(y-1)^4}$ for $(x, y) \neq (0, 1)$

then for $y \neq 1$ $g(0, y) = 0$ and for $x \neq 0$ $g(x, 1) = 0$

try a line through that point $y - 1 = mx, m \neq 0$

for $x \neq 0$ $f(x, mx+1) = \frac{mx^2}{3x^2 + 2m^4x^4} = \frac{m}{3 + 2m^4x^2} \rightarrow \frac{m}{3} \neq 0$

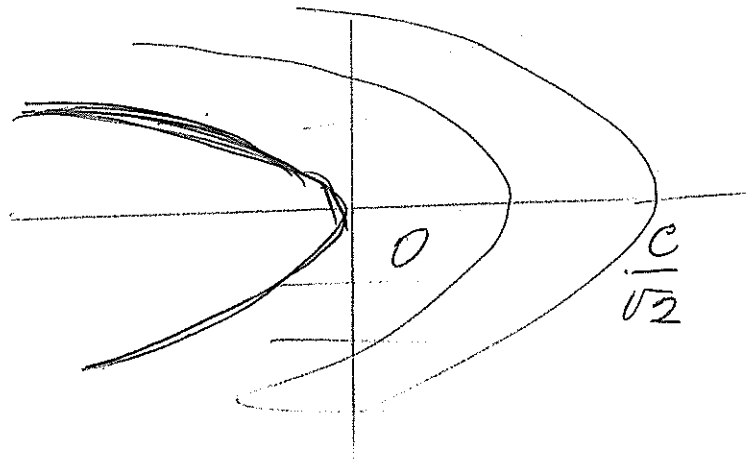
for any $m \neq 0$ (as $x \rightarrow 0$)

Therefore the limit DNE.

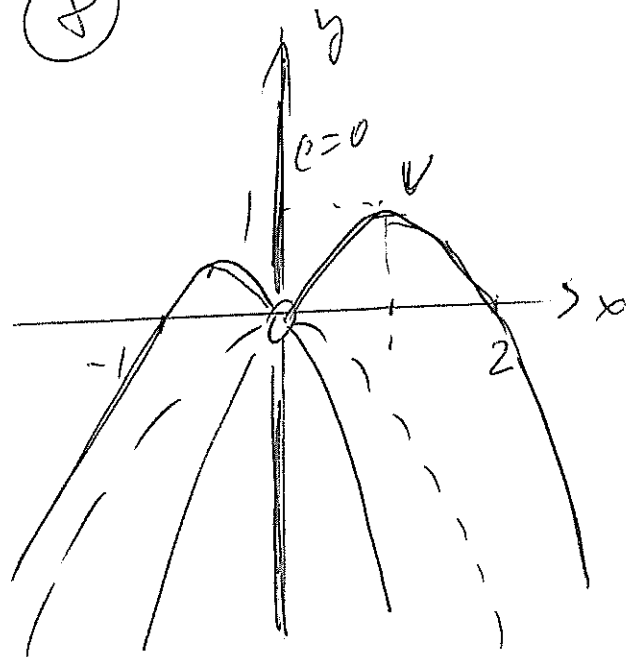
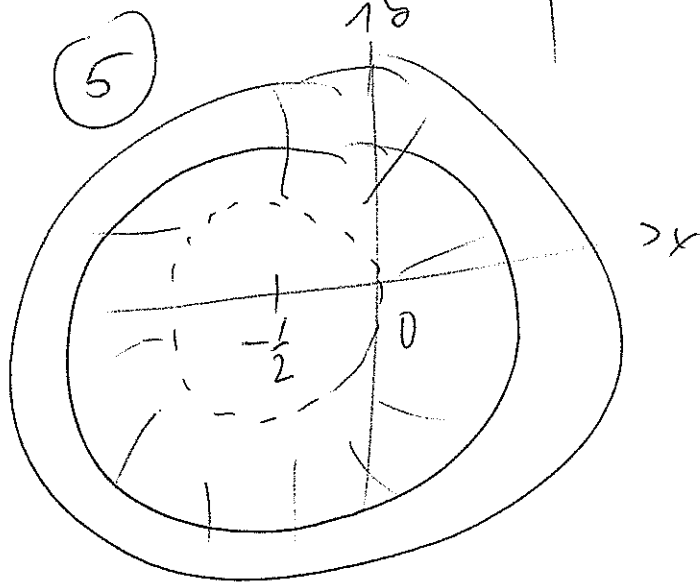
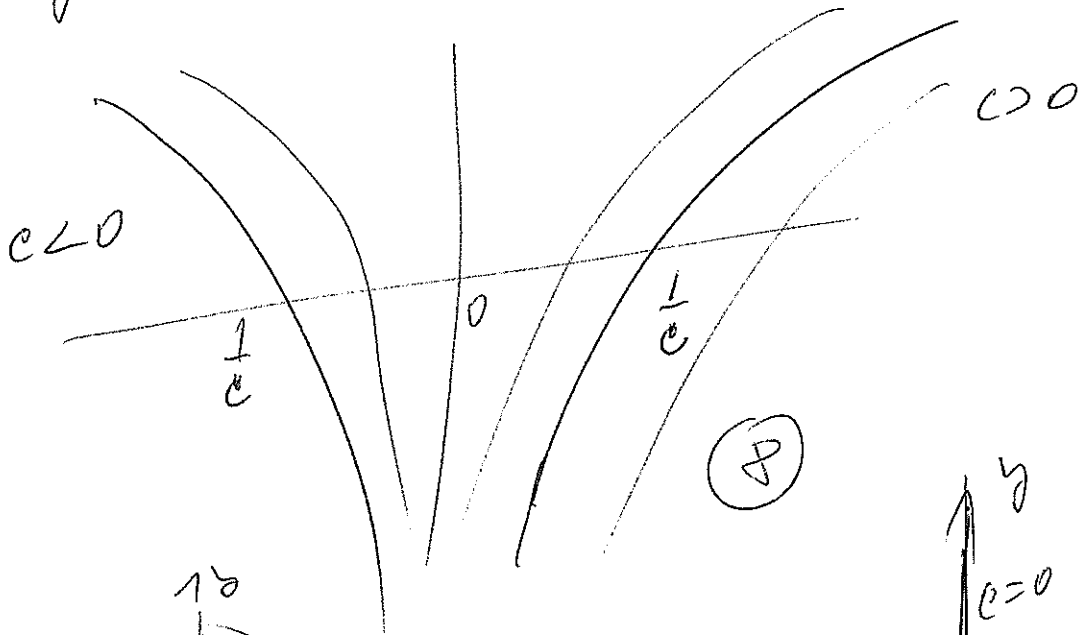
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(1) $y^2 = -2x$

$y^2 = c^2 - 2x$
 $c > 0$



(3) $y = \ln cx, cx > 0$



$y = -x^2$