

**The University of Calgary**  
**Department of Mathematics and Statistics**  
**MATH 349      Handout # 5**

**Solutions**

**For 1 a)**

For  $f(x, y) = \frac{xy}{\sqrt{1+x^2}}$        $f_y = \frac{x}{\sqrt{1+x^2}}$  and

$$f_x = y \cdot \frac{\sqrt{1+x^2} - x \frac{2x}{2\sqrt{1+x^2}}}{(1+x^2)} = y \frac{(1+x^2) - x^2}{\sqrt{1+x^2}(1+x^2)} = \frac{y}{(1+x^2)^{\frac{3}{2}}}$$

**For 1b)**

for  $s = 0$  and  $t = -1$        $x = \cos(0) = 1, y = \sin 0 = 0$

and  $\nabla f(1, 0) = \left(0, \frac{1}{\sqrt{2}}\right)$

for  $x = \cos(\pi st)$        $y = \sin \frac{\pi s}{t}$        $\frac{\partial x}{\partial s} = -\sin(\pi st) \cdot \pi t$        $\frac{\partial y}{\partial s} = \cos \frac{\pi s}{t} \cdot \frac{\pi}{t}$

and  $\frac{\partial x}{\partial s}(0, -1) = 0$        $\frac{\partial y}{\partial s}(0, -1) = -\pi$

$$\frac{\partial}{\partial s} h(0, -1) = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} = \nabla f(1, 0) \bullet \left(\frac{\partial x}{\partial s}(0, -1), \frac{\partial y}{\partial s}(0, -1)\right) = \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0, -\pi) = \frac{-\pi}{\sqrt{2}}$$

$$\frac{\partial x}{\partial t} = -\sin(\pi st) \cdot \pi s$$

$$\frac{\partial y}{\partial t} = \cos \frac{\pi s}{t} \cdot \frac{-\pi s}{t^2}$$

and  $\frac{\partial x}{\partial t}(0, -1) = 0$        $\frac{\partial y}{\partial t}(0, -1) = 0$

$$\frac{\partial}{\partial t} h(0, -1) = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = \left(0, \frac{1}{\sqrt{2}}\right) \bullet (0, 0) = 0$$

OR by Chain Rule  $Dh = Df Dg$       matrix multiplication where

$$Df = [f_x \ f_y] = \left[0 \ \frac{1}{\sqrt{2}}\right] \text{ at } x = 1, y = 0$$

$$Dg = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \text{ at } s = 0, t = -1 \quad Dg(0, -1) = \begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix}$$

$$\text{so } Dh = [h_s \ h_t] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\pi & 0 \end{bmatrix} = \begin{bmatrix} -\pi & 0 \\ \sqrt{2} & 0 \end{bmatrix}$$

**For 2)**

for  $f(x, y) = \ln(x + y^2)$  and  $x_0 = 0, y_0 = -1$        $z_0 = f(0, -1) = \ln 1 = 0$

partials  $f_x = \frac{1}{x+y^2}$        $f_y = \frac{2y}{x+y^2}$       and  $A = f_x(0, -1) = 1, B = f_y(0, -1) = -2$

so tangent plane  $z = z_0 + A(x - x_0) + B(y - y_0)$

$$z = 0 + (x - 0) - 2(y + 1) \quad z = x - 2y - 2$$

Or the point on the graph is  $P(0, -1, 0)$  and  $\mathbf{n} = (\nabla f, -1) = (1, -2, -1)$

so  $x - 2y - z = d$  and from  $P$        $0 + 2 - 0 = d$  thus  $x - 2y - z = 2$ .

**For 3)**

for  $f(x, y) = e^{\sqrt{\frac{y}{x}}}$  the domain of  $f$  is  $\frac{y}{x} \geq 0$  i.e.  $\{y \geq 0, x > 0\} \cup \{y \leq 0, x < 0\}$   
 but the domain of the partials is  $\{y > 0, x > 0\} \cup \{y < 0, x < 0\} = \{xy > 0\}$

$$f_x = e^{\sqrt{\frac{y}{x}}} \cdot \frac{1}{2} \left(\frac{y}{x}\right)^{-\frac{1}{2}} \cdot \frac{-y}{x^2} = -\frac{\sqrt{y}}{2x^{\frac{3}{2}}} e^{\sqrt{\frac{y}{x}}} \text{ for } x > 0, y > 0 \text{ only since } \sqrt{\frac{y}{x}} = \frac{\sqrt{y}}{\sqrt{x}}$$

$$f_{xx} = e^{\sqrt{\frac{y}{x}}} \left[ \left(-\frac{\sqrt{y}}{2} x^{-\frac{3}{2}}\right)^2 + \frac{3}{4} \sqrt{y} x^{-\frac{5}{2}} \right] = e^{\sqrt{\frac{y}{x}}} \left[ \frac{y}{4x^3} + \frac{3}{4x^2} \sqrt{\frac{y}{x}} \right]$$

$$f_{xy} = -\frac{1}{2}x^{-\frac{3}{2}}e^{\sqrt{\frac{y}{x}}}\left[\frac{1}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{y}\sqrt{x}}\right] = -\frac{1}{4}e^{\sqrt{\frac{y}{x}}}\cdot\left(\frac{1}{x\sqrt{yx}} + \frac{1}{x^2}\right), \text{ for } xy > 0$$

the middle steps are true only if both  $x, y$  are positive  
but the final expressions are valid even if both are negative.

**For 4a)**

for  $z > 0$   $\nabla f = \text{grad}f = (f_x, f_y, f_z)$  where

$$f_x = \sqrt{2}\cos(\pi xy + x \ln z) [\pi y + \ln z] \quad f_y = \sqrt{2}\cos(\pi xy + x \ln z) [\pi x]$$

$$f_z = \sqrt{2}\cos(\pi xy + x \ln z) \left[\frac{x}{z}\right] \text{ so } \nabla f = \sqrt{2}\cos(\pi xy + x \ln z) \left(\pi y + \ln z, \pi x, \frac{x}{z}\right)$$

**For 4b)**

$$\mathbf{g}'(t) = (g'_1(t), g'_2(t), g'_3(t)) = \left(-\frac{1}{t^2}, \frac{1}{t^2}, \frac{1}{2}\right) \text{ for } t \neq 0$$

OR

$$D\mathbf{g} = \begin{bmatrix} g'_1 \\ g'_2 \\ g'_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{t^2} \\ \frac{1}{t^2} \\ \frac{1}{2} \end{bmatrix}, \text{ where } g_1(t) = \frac{1}{t}, g_2(t) = -\frac{1}{t}, \text{ and } g_3(t) = \frac{t}{2}$$

**For 4c)**

if  $t = 2$  then  $x = \frac{1}{2}, y = -\frac{1}{2}$  and  $z = 1$   $\sqrt{2}\cos(\pi xy + x \ln z) = \sqrt{2}\cos\frac{-\pi}{4} = 1$   
and by Chain Rule

$$Dh = Df D\mathbf{g}$$

$$Df = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} -\frac{\pi}{2} & \frac{\pi}{2} & \frac{1}{2} \end{bmatrix} \text{ at } \left(\frac{1}{2}, -\frac{1}{2}, 1\right), \text{ and } \mathbf{g}'(2) = \left(\frac{-1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

$$h'(2) = Df\left(\frac{1}{2}, -\frac{1}{2}, 1\right)D\mathbf{g}(2) = \begin{bmatrix} -\frac{\pi}{2} & \frac{\pi}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \frac{\pi}{8} + \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+1}{4}.$$

OR

$$h'(2) = f_x x' + f_y y' + f_z z' = \nabla f\left(\frac{1}{2}, -\frac{1}{2}, 1\right) \bullet (g'_1(2), g'_2(2), g'_3(2)) =$$

$$= \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{1}{2}\right) \bullet \left(\frac{-1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{\pi+1}{4}$$

**For 5)**

Simplify the function first  $f(x, y) = \ln \frac{x^2 + y^2}{xy} = \ln(x^2 + y^2) - \ln x - \ln y$

for  $x > 0, y > 0$  so

$$\text{partials are } f_x = \frac{2x}{x^2 + y^2} - \frac{1}{x} \quad f_y = \frac{2y}{x^2 + y^2} - \frac{1}{y}$$

$$\text{and } f_x(2, 1) = \frac{4}{5} - \frac{1}{2} = \frac{3}{10} \quad f_y(2, 1) = \frac{2}{5} - 1 = \frac{-3}{5}$$

Now, rate of change means the directional derivative

Since partials are continuous in the domain

$$D_\nu f(2, 1) = \nabla f(2, 1) \bullet (\nu_1, \nu_2) = \left(\frac{3}{10}, -\frac{3}{5}\right) \bullet (\nu_1, \nu_2) = \frac{3}{10}(\nu_1 - 2\nu_2)$$

and we are looking for a unit vector  $(\nu_1, \nu_2)$  such that

$$D_\nu f(2, 1) = \frac{3}{10} \quad \frac{3}{10}(\nu_1 - 2\nu_2) = \frac{3}{10} \quad \text{and } \nu_1^2 + \nu_2^2 = 1$$

so  $\nu_1 - 2\nu_2 = 1$ ; one solution is  $\nu_1 = 1, \nu_2 = 0$ ,

$$\text{generally } \nu_1 = 1 + 2\nu_2 \implies \nu_1^2 + \nu_2^2 = 5\nu_2^2 + 4\nu_2 + 1 = 1,$$

$\nu_2(5\nu_2 + 4) = 0$  thus another solution is  $\nu_2 = -\frac{4}{5}$  and  $\nu_1 = 1 - \frac{8}{5} = -\frac{3}{5}$   
 Together  $\boldsymbol{\nu} = (1, 0)$  or  $\boldsymbol{\nu} = \frac{1}{5}(-3, -4)$

Maximum rate is always equal to  $\|\nabla f\| = \left\| \left( \frac{3}{10}, -\frac{3}{5} \right) \right\| = \frac{3}{10}\sqrt{1+4} = \frac{3\sqrt{5}}{10}$ .

**For 6)**

$\nabla f = (f_x, f_y, f_z) = (2xze^y + z^2, x^2ze^y, x^2e^y + 2xz)$  at  $P$

$\nabla f(P) = (12, 4, 6) = 2(6, 2, 3)$   $\|\nabla f\| = 2\sqrt{49} = 14$

the direction is  $\mathbf{u} = -\frac{\nabla f}{\|\nabla f\|} = -\frac{1}{7}(6, 2, 3)$  and rate is  $\|\nabla f\| = 14$ .

**For 7a)**

$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{2x^2 + y^2}$  does not exist, for sure it is NOT equal to 0

since for  $x \neq 0$   $f(x, x) = \frac{1}{3}$  so  $f$  is **discontinuous** at  $(0, 0)$ ;

**For 7b)**

by definition  $f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - 0}{x} = 0$

$f_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - 0}{y} = 0$  so the gradient exists and  $\nabla f(0, 0) = (0, 0)$ .

**For 7c)**

for the directional derivative we have to use the definition

since  $f$  is discontinuous at  $(0, 0)$  thus partials cannot be continuous at  $(0, 0)$

unit vector in the direction of the line  $y = x$   $x = t, y = t$  is  $\mathbf{u} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

$D_{\mathbf{u}}f(0, 0) = \frac{1}{\sqrt{2}} \lim_{t \rightarrow 0} \frac{f(t, t) - 0}{t} = \frac{1}{\sqrt{2}} \lim_{t \rightarrow 0} \frac{1}{t} \cdot \frac{t^2}{3t^2} = \lim_{t \rightarrow 0} \frac{1}{3t}$  does not exist

OR from a)

since  $f(x, x) = \frac{1}{3}$  for  $x \neq 0$  and  $f(0, 0) = 0$   $f$  is not continuous along that line  
 so it cannot be differentiable along that line.

**For 7d)**

for any other point we can use the theorem since partials are continuous

$D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}$

first  $f_x = y \frac{2x^2 + y^2 - 4x^2}{(2x^2 + y^2)^2} = y \frac{y^2 - 2x^2}{(2x^2 + y^2)^2}$

and  $f_y = x \frac{2x^2 + y^2 - 2y^2}{(2x^2 + y^2)^2} = x \frac{2x^2 - y^2}{(2x^2 + y^2)^2}$  so at the given point

$\nabla f(-1, -1) = \left( \frac{1}{9}, -\frac{1}{9} \right)$  and  $D_{\mathbf{u}}f(-1, -1) = \frac{1}{9\sqrt{2}}(1, -1) \bullet (1, 1) = 0$