

FINAL HANDOUT
MATH 349 SOLUTION .

1. For $a_n = \frac{n^3}{3^n}$ $a_n > 0$, so not alternating; define $f(x) = \frac{x^3}{3^x}$ for $x > 0$

$$\text{then } \lim_{x \rightarrow \infty} \frac{x^3}{3^x} = \text{"}\infty\text{" (L'H.R.-3 times)} = \lim_{x \rightarrow \infty} \frac{3x^2}{3^x \ln 3} = \dots \lim_{x \rightarrow \infty} \frac{6}{3^x (\ln 3)^3} = \text{"}\frac{6}{\infty}\text{"} = 0$$

the sequence is convergent therefore bounded

$$f'(x) = \left(\frac{x^3}{3^x}\right)' = \frac{3x^2 3^x - x^3 3^x \ln 3}{(3^x)^2} = \frac{x^2(3 - x \ln 3)}{3^x} < 0 \text{ if } x > \frac{3}{\ln 3} = 2.73$$

thus the sequence is ult.decr. for $n \geq 3$

$a_1 = \frac{1}{3} < a_2 = \frac{8}{9} < a_3 = 1$ so an upper bound is 1, a lower bound is 0.

2. for
$$\sum_{n=1}^{\infty} \frac{(3x-1)^n}{3^{3n} \sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n (x - \frac{1}{3})^n}{3^{3n} \sqrt{n}}.$$

the centre is $c = \frac{1}{3}$ and the coefficients $a_n = \frac{1}{3^{2n} \sqrt{n}}$

the radius of convergence $R = \frac{1}{L} = 9$

$$\text{since } L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3^{2n} \sqrt{n}}{3^{2n+2} \sqrt{n+1}} = \frac{1}{9} \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = \frac{1}{9}$$

thus the series is absolutely convergent on $(\frac{1}{3} - 9, \frac{1}{3} + 9) = (\frac{-26}{3}, \frac{28}{3})$

ends: $x = \frac{1}{3} + 9$ gives $\sum_{n=1}^{\infty} \frac{(x - \frac{1}{3})^n}{3^{2n} \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent p-series

$p = \frac{1}{2} < 1$

for $x = \frac{1}{3} - 9$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is cond.convergent since the sequence $\frac{1}{\sqrt{n}} \searrow 0$

(positive, decr. since $\frac{1}{\text{incr.}}$, and limit is $\frac{1}{\infty} = 0$)

together the interval of convergence is $[\frac{-26}{3}, \frac{28}{3})$, outside the series is divergent.

3. we know that $e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!}$ for any s , use $s = x + 1$

$$\begin{aligned} \text{then } f(x) = xe^x &= [(x+1) - 1]e^{x+1}e^{-1} = \frac{1}{e}(x+1) \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} - \\ &\frac{1}{e} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \\ &= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{n!} - \frac{1}{e} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \frac{1}{e} \sum_{k=1}^{\infty} \frac{(x+1)^k}{(k-1)!} - \frac{1}{e} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} = \\ &= -\frac{1}{e} + \frac{1}{e} \sum_{k=1}^{\infty} \left[\frac{1}{(k-1)!} - \frac{1}{k!} \right] (x+1)^k = -\frac{1}{e} + \frac{1}{e} \sum_{k=1}^{\infty} \frac{k-1}{k!} (x+1)^k \text{ for} \\ &\text{any } x. \end{aligned}$$

check $a_0 = f(-1) = -e^{-1}$.

also: you can use the definition:

$$\begin{aligned} a_0 &= f(-1) = -e^{-1} & a_1 &= f'(-1) = [e^x + xe^x]_{x=-1} = 0 \\ a_2 &= \frac{1}{2} f''(-1) = \frac{1}{2} [2e^x + xe^x]_{x=-1} = \frac{2-1}{2e} = \frac{1}{2e} \\ a_3 &= \frac{1}{3!} f'''(-1) = \frac{1}{6} [3e^x + xe^x]_{x=-1} = \frac{3-1}{6e} = \frac{1}{3e} \end{aligned}$$

we can see that $f^{(n)}(-1) = [ne^x + xe^x]_{x=-1} = \frac{n-1}{e}$ and $a_n = \frac{n-1}{en!}$
as above

4. For the curve $c = \{z = x^2 + y^2\} \cap \{6x - 2y - z = 1\}$

for a) $z = x^2 + y^2 = 6x - 2y - 1$ so $x^2 + y^2 - 6x + 2y + 1 = 0$, complete the squares

$$(x-3)^2 + (y+1)^2 = 9 \text{ and } \left(\frac{x-3}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1 \text{ so}$$

$$x = 3 + 3 \cos t, y = -1 + 3 \sin t, z = 6x - 2y - 1 = 19 + 18 \cos t - 6 \sin t, t \in [0, 2\pi)$$

for b) we have two options to find \vec{d} in $(x, y, z) = (0, -1, 1) + t \vec{d}$

from part a:

$$t = \pi \text{ for the point } P = \vec{r}(\pi) \text{ and } \vec{d} = \vec{r}'(\pi) = (0, -3, 6) \text{ or } (0, 1, -2)$$

since $\vec{r}'(t) = (-3 \sin t, 3 \cos t, -18 \sin t - 6 \cos t)$

OR

$\vec{d} = \vec{n}_1 \times \vec{n}_2$ where $\vec{n}_1 = \nabla F = (2x, 2y, -1)$ then at P $\vec{n}_1 = (0, -2, -1)$

where $F(x, y, z) = x^2 + y^2 - z = 0$ and $G(x, y, z) = 6x - 2y - z = 1$
and $\vec{n}_2 = \nabla G = (6, -2, -1)$ $\vec{n}_1 \times \vec{n}_2 = (0, 6, -12) = 6(0, 1, -2)$
so $(x, y, z) = (0, -1, 1) + t(0, 1, -2)$.

for c) for P $t = \pi$ and for R $t = \frac{\pi}{2}$, also

$$\|\vec{r}'(t)\| = \sqrt{9 + 36(3 \sin t + \cos t)^2} = \sqrt{45 + 36(8 \sin^2 t + 6 \sin t \cos t)}$$

$$\text{so } s = \int_{\frac{\pi}{2}}^{\pi} \sqrt{45 + 72(4 \sin^2 t + 3 \sin t \cos t)} dt.$$

5. the equation $z = f(1, -1) + \nabla f(1, -1) \bullet (x - 1, y + 1) = 1 - (x - 1)$ gives

$$z = 1 - (x - 1) = 2 - x$$

$$\text{since for } f(x, y) = e^{yx^2 \ln x} \quad f(1, -1) = 1,$$

$$f_x(x, y) = e^{yx^2 \ln x} (2xy \ln x + yx) \quad f_x(1, -1) = -1$$

$$f_y(x, y) = e^{yx^2 \ln x} x^2 \ln x \quad f_y(1, -1) = 0$$

OR

$\vec{n} = (\nabla f(1, -1), -1) = (-1, 0, -1)$ and the point is $(1, -1, f(1, -1)) = (1, -1, 1)$

so $-x - z = d$ and through $P(1, -1, 1)$ $x + z = 2$.

6. **for a)** at the origin we have to use the definition:

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{1 - 0}{x} = \lim_{x \rightarrow 0^{\pm}} \frac{1}{x} \quad DNE (\pm\infty)$$

$$f_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

thus the gradient does not exist at $(0, 0)$.

for b) since $f(x, x) = \frac{1}{4}$ for any $x \neq 0$ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) \neq 0 = f(0, 0)$

the function is discontin. at $(0, 0)$; (the limit *DNE*).o

7. .

for the domain $\frac{x}{y} > 0$ both positive OR both negative

so the domain consists of the first and third quadrants without the axes
for level curves for any c

$c = \ln \frac{x}{y}, e^c = \frac{x}{y}, y = e^{-c}x$ lines through the origin

without the origin and with positive slopes $m = e^{-c} = 1, e, \frac{1}{e}$

since we got a level curve for any c the range is $z \in (-\infty, \infty)$.

8. $\nabla f = (2xe^{-y}, -e^{-y}(x^2 + \cos z), -e^{-y} \sin z)$ all partials are cont. functions
so

$D_v f = \nabla f \bullet v$ where $\nabla f(A) = (4, -3, 0)$

v is the unit vector in the direction of $\overrightarrow{AB} = (-3, -1, -\pi)$

and $\|\overrightarrow{AB}\| = \sqrt{10 + \pi^2}$ and finally

$$D_v f(A) = \frac{1}{\sqrt{10 + \pi^2}} (4, -3, 0) \bullet (-3, -1, -\pi) = \frac{-9}{\sqrt{10 + \pi^2}}.$$

9. Since $x = uv^2, y = \frac{u}{v}$. for $u = 2, v = -1$

$x = 2$ $y = -2$ so we need $f_x(2, -2) = 3, f_y(2, -2) = -2$

$$\frac{\partial F}{\partial u}(u, v) = \nabla f(x, y) \bullet \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = (f_x, f_y) \bullet \left(v^2, \frac{1}{v}\right)$$

$$\text{and } \frac{\partial F}{\partial u}(2, -1) = (3, -2) \bullet (1, -1) = 5.$$

10. **for a)**

for $T = 0$ $y^2 - 3x^2 = 0$ $y = \pm x\sqrt{3}$ two lines

for $T = 3$ $y^2 - 3x^2 = 3$ hyperbola with intercepts $y = \pm\sqrt{3}, x = 0$

for $T = 6$ $y^2 - 3x^2 = -6$ hyperbola with intercepts $x = \pm\sqrt{2}, y = 0$

for b)

the direction is $-\nabla T(-1, 2)$

so first $\nabla T = (-6x, 2y)$ then $\nabla T(-1, 2) = (6, 4) = 2(3, 2)$ so $\mathbf{u} = \frac{-1}{\sqrt{13}}(3, 2)$.

for c)

the direction must be perpendicular to $\nabla T(-1, 2) = 2(3, 2)$

so $\mathbf{u} = \frac{\pm 1}{\sqrt{13}}(2, -3)$

11. Define $F_1(x, y, z) = xy^2 - z + u^2 - 3, F_2(x, y, z) = x^3z + 2y - u$
 $F_3(x, y, z) = xu + y - xyz - 3$.

since **all partials are continuous function** the only condition is that

$$\left\| \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \right\| = \det \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} \neq 0 \text{ at the given point}$$

$$\text{so det} \begin{bmatrix} y^2 & 2xy & -1 \\ 3x^2z & 2 & x^3 \\ u - yz & 1 - xz & -xy \end{bmatrix} \text{ at the given point} = \det \begin{bmatrix} 1 & 2 & -1 \\ -3 & 2 & 1 \\ 2 & 2 & -1 \end{bmatrix} =$$

along the first column

$$= 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} - (-3) \cdot \det \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = -4 + 0 + 8 = 4 \neq 0.$$

12. For $f(x, y) = \ln(y \cos x)$ around $x = 0, y = 1$

$$T_2(x, y) = f(0, 1) + \nabla f(0, 1) \bullet (x, y - 1) + \frac{1}{2} [Ax^2 + 2Bx(y - 1) + C(y - 1)^2]$$

where $f(0, 1) = \ln 1 = 0$ since $f(x, y) = \ln y + \ln \cos x$ for $y > 0, \cos x > 0$

$$f_x = \frac{-\sin x}{\cos x} = -\tan x \quad f_y = \frac{1}{y} \quad \nabla f(0, 1) = (0, 1)$$

$$f_{xx} = -\sec^2 x = \frac{-1}{\cos^2 x} \quad f_{xy} = 0 \quad f_{yy} = -\frac{1}{y^2}$$

$$\text{and} \quad A = f_{xx}(0, 1) = -1, B = f_{xy} = 0 \quad C = f_{yy}(0, 1) = -1$$

thus

$$\begin{aligned} T_2(x, y) &= 0 + (0, 1) \bullet (x, y - 1) + \frac{1}{2} [-x^2 + 0x(y - 1) - (y - 1)^2] = \\ &= y - 1 - \frac{x^2}{2} - \frac{(y - 1)^2}{2}. \end{aligned}$$

13. Define $F(x, y, z) = \arcsin(zy) + z^3x + x^2y + 8$ for $-1 \leq zy \leq 1, x$ any

$$\text{then } \frac{\partial F}{\partial x} = z^3 + 2xy \quad \frac{\partial F}{\partial y} = \frac{z}{\sqrt{1 - z^2y^2}} + x^2 \text{ and } \frac{\partial F}{\partial z} = \frac{y}{\sqrt{1 - z^2y^2}} + 3z^2x$$

for $-1 < zy < 1, x$ any

all functions are cont. at the point $P(-1, 0, 2)$ and $\frac{\partial F}{\partial z}(P) = -12 \neq 0$

so the equation

can be solved for z as a function of x, y around the point $P(-1, 0, 2)$

$$\text{then } \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2xy + z^3}{\frac{y}{\sqrt{1 - z^2y^2}} + 3z^2x} = \frac{-8}{-12} = \frac{2}{3} \text{ at } P \text{ and}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{\frac{z}{\sqrt{1 - z^2y^2}} + x^2}{\frac{y}{\sqrt{1 - z^2y^2}} + 3z^2x} = -\frac{3}{-12} = \frac{1}{4} \text{ at that point..}$$