DEPARTMENT OF MATHEMATICS AND STATISTICS

MIDTERM EXAM

MATH 349 LEC 01/02

Fall 2008 **TIME:90 minutes**

SOLUTION
Each questions is for 10 marks.Calculators allowed.Table attached. **Total 60**
Explain each step.

1. the sequence $a_n = \dfrac{\sqrt{n}}{n^2+27} > 0$...a lower bound,also $a_n < 1$ ..an upper bound

   $$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+27} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^{\frac{3}{2}}}}{1+\frac{27}{n^2}} = 0 \text{...convergent}$$

   $$f'(x) = \left(\frac{\sqrt{x}}{x^2+27}\right)' = \frac{\frac{1}{2\sqrt{x}}\left(x^2+27\right) - 2x\sqrt{x}}{(x^2+27)^2} = \frac{\left(x^2+27\right) - 4x^2}{2\sqrt{x}(x^2+27)^2} = \frac{27-3x^2}{2\sqrt{x}(x^2+27)^2} < 0$$

   for $x > 3$

   thus the sequence is ult.decreasing and the best upper bound is $a_3 = \dfrac{\sqrt{3}}{36}$.

2. For $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n} (x-1)^n$ the centre, is $c = 1, a_n = \dfrac{\sqrt{\ln n}}{n}$

   for the radius :$\dfrac{a_{n+1}}{a_n} = \dfrac{\sqrt{\ln(n+1)}}{n+1} \cdot \dfrac{n}{\sqrt{\ln n}} = \sqrt{\dfrac{\ln(n+1)}{\ln n}} \cdot \dfrac{n}{n+1} \to 1$

   since $\displaystyle\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = (\text{ L'H.R.}) = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = 1$ $\qquad R = 1$

   and the series is abs.convergent on $(0, 2)$

   for $x = 2$ $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n}$ is divergent by Integral Test: $\displaystyle\int_2^{\infty} \frac{\sqrt{\ln x}}{x} dx = \left[\frac{2}{3}(\ln x)^{\frac{3}{2}}\right]_2^{\infty} = \infty$

   since $\displaystyle\lim_{x\to\infty} \ln x = +\infty$
   and also

   $f(x) = \dfrac{\sqrt{\ln x}}{x}$ is decreasing since $f'(x) = \left(\dfrac{\sqrt{\ln x}}{x}\right)' = \dfrac{\frac{1}{2\sqrt{\ln x}} - \sqrt{\ln x}}{x^2} = \dfrac{1-2\ln x}{2x^2\sqrt{\ln x}} < 0$ for $x \geq 2$

   for $x = 0$ $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{\ln n}}{n} (-1)^n$ is cond.convergent by Alt.Test since from above the sequence $a_n = \dfrac{\sqrt{\ln n}}{n}$

   is decreasing and $\displaystyle\lim_{n\to\infty} \frac{\sqrt{\ln n}}{n} = \lim_{x\to\infty} \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = 0$

   and interval of cinvergence is $[0, 2)$.

3. $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$.by Intergral Test:

$f(x) = \frac{1}{x}$ is decr.and positve for $x \geq 1, f'(x) = \frac{-1}{x^2} < 0$

and $\int_1^{\infty}\frac{1}{x}dx = \lim_{x\to\infty} \ln x - \ln 1 = +\infty$

4. Find the sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!} - (n=0) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!} - (-1) = 1 - \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{(2n)!} = 1 - \cos 2.$

5. the Taylor series for $f(x) = x + e^x = \sum_{n=0}^{\infty} a_n(x+1)^n$

set $t = x + 1 \qquad x = t - 1$

$x + e^x = t - 1 + e^{t-1} = -1 + t + \frac{1}{e}\sum_{n=0}^{\infty}\frac{t^n}{n!} = -1 + t + \frac{1}{e} + \frac{1}{e}t + \frac{1}{e}\sum_{n=2}^{\infty}\frac{t^n}{n!} =$

$= (-1 + \frac{1}{e}) + t\left(1 + \frac{1}{e}\right) + \sum_{n=2}^{\infty}\frac{1}{en!}(x+1)^n$ for any $x., a_n = \frac{1}{en!}$ for $n \geq 2$

ALSO

we know that $a_0 = f(-1) = -1 + \frac{1}{e}$ and $a_1 = f'(-1) = 1 + \frac{1}{e}$

since $f'(x) = 1 + e^x$ and all $f^{(n)}(x) = e^x$ so $a_n = \frac{e^{-1}}{n!}$ for $n \geq 2$.

6. For $\quad \mathbf{r}(t) = (\cos(2t), \sin(2t), 3t^2) \qquad \mathbf{r}'(t) = (-2\sin(2t), 2\cos(2t), 6t)$

   (a) for the point $P(1,0,0) \qquad t = 0 \qquad \mathbf{d} = \mathbf{r}'(0) = (0,2,0)$ or $(0,1,0)$
   and the tangent is given by
   $(x,y,z) = (1,0,0) + t(0,1,0)$ OR $x = 1, y = t, z = 0$

   (b) for the point $R\left(-1, 0, \frac{3}{4}\pi^2\right) \qquad t = \frac{\pi}{2}$
   $\|\mathbf{r}'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 36t^2} = \sqrt{4 + 36t^2} = 2\sqrt{1 + 9t^2}$
   then
   $l = \int_0^{\frac{\pi}{2}} \|\mathbf{r}'(t)\| dt = 2\int_0^{\frac{\pi}{2}} \sqrt{1+9t^2}dt = (\text{subst.}u = 3t, du = 3dt) = \frac{2}{3}\int_0^{\frac{3}{2}\pi}\sqrt{1+u^2}du =$
   $= \frac{1}{3}\left[u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2}\right]_0^{\frac{3}{2}\pi} = \frac{\pi}{2}\sqrt{1 + \frac{9}{4}\pi^2} + \frac{1}{3}\ln(\frac{3}{2}\pi + \sqrt{1 + \frac{9}{4}\pi^2}).$