

MATH 349  
Midterm Handout-Solution

Question #1.

For the sequence  $a_n = \frac{\ln(n+3)}{n+3}$

1.  $\lim_{n \rightarrow \infty} a_n = \text{"}\infty\text{" } L'H.R. = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+3}}{1} = \text{"}\frac{1}{\infty}\text{"} = 0$

so the sequence is **convergent and thus bounded**;

For monotonicity, define  $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \text{ for } \ln x > 1, x > e,$$

thus the sequence is **decreasing** for  $x = n + 3 \geq 3, n \geq 1$

and an lower bound is 0, an upper bound is  $a_1 = \frac{\ln 4}{4}$ .

Question #2.

First abs.convergence  $\sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \dots =$

$$= 1 - \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 1 - 0 = 1 \text{ (It is a telescoping series.)}$$

$$\sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \text{ is absolutely convergent .}$$

Question #3.

For  $\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{k}} (x-1)^k, c=1, a_k = \frac{2^k}{\sqrt{k}}$

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{2^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{2^k} = \sqrt{\frac{k}{k+1}} \cdot 2 \rightarrow 2 \quad R = \frac{1}{2}$$

and the series is abs.convergent on  $\left(\frac{1}{2}, \frac{3}{2}\right)$

For the ends  $x = \frac{1}{2}$  or  $\frac{3}{2}$  we get  $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$  or  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

the second one is divergent since  $p = \frac{1}{2} < 1$ , but the first one is cond.convergent since the sequence  $\frac{1}{\sqrt{k}} \searrow 0$

Together the series is convergent on  $\left[\frac{1}{2}, \frac{3}{2}\right)$

Question #4.

For a)  $\sum_{k=1}^{\infty} \frac{(\ln 2)^k}{k!}$  do not mix  $(\ln 2)^k$  and  $\ln 2^k = k \ln 2$

we know that  $e^s = \sum_{k=0}^{\infty} \frac{s^k}{k!}$  for any  $s$  so  $e^s - 1 = \sum_{k=1}^{\infty} \frac{s^k}{k!}$

and for  $s = \ln 2$  we get

$$\sum_{k=1}^{\infty} \frac{(\ln 2)^k}{k!} = e^{\ln 2} - 1 = 2 - 1 = 1.$$

**For b)**  $\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n(n+1)}$  we need to know the sum of  $\sum_{n=3}^{\infty} \frac{(x)^n}{(n+1)}$  for  $x = -\frac{1}{2}$

and for  $x \neq 0$

$$\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = \frac{1}{x} \sum_{n=3}^{\infty} \frac{x^{n+1}}{(n+1)} = \frac{1}{x} \left[ \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)} - \frac{x^3}{3} - \frac{x^2}{2} - \frac{x}{1} \right]$$

and since we know that  $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)}$

$$\sum_{n=3}^{\infty} \frac{x^n}{(n+1)} = -\frac{1}{x} \ln(1-x) - \frac{x^2}{3} - \frac{x}{2} - 1 \text{ and finally for } x = -\frac{1}{2}$$

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{2^n(n+1)} = 2 \ln \frac{3}{2} - \frac{1}{12} + \frac{1}{4} - 1 = 2 \ln \frac{3}{2} - \frac{1-3+12}{12} = 2 \ln \frac{3}{2} - \frac{5}{6}.$$

## Question#5.

USE Partial fraction first , then  $x+1 = t, x = t-1$

$$\begin{aligned} \text{for } x \neq 0, -3 \quad \frac{1}{(x+3)x} &= \frac{1}{3} \left[ \frac{-1}{x+3} + \frac{1}{x} \right] = \frac{1}{3} \left[ \frac{-1}{t+2} + \frac{1}{t-1} \right] = \\ &= \frac{1}{3} \left[ \frac{-1}{2} \cdot \frac{1}{1+\frac{t}{2}} - \frac{1}{1-t} \right] = \end{aligned}$$

$$\left( \text{using } \frac{1}{1+r} = \sum_{n=0}^{\infty} (-r)^n \text{ for } r = \frac{t}{2} \text{ and } \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \text{ for } r = t \right)$$

$$= -\frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \left(\frac{t}{2}\right)^n - \frac{1}{3} \sum_{n=0}^{\infty} t^n = \frac{-1}{3} \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{2^{n+1}} + 1 \right] (x+1)^n$$

for  $-1 < \frac{x+1}{2} < 1$   $-2 < x+1 < 2$  and  $-1 < x+1 < 1$ , together

$$-2 < x < 0. \text{ And } a_6 = -\frac{1}{3} \cdot \left[ \frac{1}{2^7} + 1 \right] = -\frac{1+2^7}{3 \cdot 2^7} = -\frac{129}{384}.$$

## Question#6.

By definition

$$a_0 = f(2) = \ln \frac{1}{2} = -\ln 2 \quad a_1 = f'(2) = \frac{1}{2}$$

$$\text{since } f'(x) = [\ln(x-1) - \ln x]' = \frac{1}{x-1} - \frac{1}{x}, \text{ then } f''(x) = \frac{-1}{(x-1)^2} + \frac{1}{x^2}$$

$$\text{and } a_2 = \frac{1}{2} f''(2) = \frac{1}{2} \left( \frac{-1}{1} + \frac{1}{4} \right) = -\frac{3}{8},$$

$$\text{finally } f'''(x) = \frac{2}{(x-1)^3} - \frac{2}{x^3} \text{ and } a_3 = \frac{1}{6} f'''(2) = \frac{1}{3} \cdot \left( 1 - \frac{1}{8} \right) = \frac{7}{24}$$

$$T_3(x) = -\ln 2 + \frac{1}{2}(x-2) - \frac{3}{8}(x-2)^2 + \frac{7}{24}(x-2)^3.$$

### Question #7.

curve  $c$  given as the intersection of

the cone  $\{z = \sqrt{2x^2 + 2y^2}\}$  and the plane  $\{z + x = 1\}$ .

from the plane  $z = 1 - x$  substitute into the cone  $(1 - x)^2 = 2x^2 + 2y^2$

$$1 - 2x + x^2 = 2x^2 + 2y^2 \quad 1 = x^2 + 2x + 2y^2 \quad \text{complete the square}$$

$$2 = (x+1)^2 + 2y^2 \quad 1 = \left(\frac{x+1}{\sqrt{2}}\right)^2 + y^2$$

and

$$\text{then a parametrization is } \frac{x+1}{\sqrt{2}} = \cos t \quad y = \sin t$$

$$x = -1 + \sqrt{2} \cos t \quad y = \sin t \text{ and}$$

$$z = 1 - x = 2 - \sqrt{2} \cos t \quad t \in [0, 2\pi].$$

### Question #8.

For the curve  $c$  given by  $\mathbf{r}(t) = (2t, t^2, \ln t)$ ,  $t > 0$

$$t = 1 \quad \text{for } P(2, 1, 0) \quad t = e \quad \text{for } R(2e, e^2, 1)$$

$$\text{find } \mathbf{r}'(t) = \left(2, 2t, \frac{1}{t}\right)$$

then **for a)**

$\mathbf{d} = \mathbf{r}'(1) = (2, 2, 1)$  and the tangent line is

$$(x, y, z) = (2, 1, 0) + s(2, 2, 1)$$

$$\text{or } x = 2 + 2s \quad y = 1 + 2s \quad z = s$$

**for b)** for arclength we need

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\left(\frac{2t^2 + 1}{t}\right)^2} = \frac{2t^2 + 1}{|t|}$$

$$\text{and } s = \int_1^e \frac{2t^2 + 1}{t} dt = \int_1^e \left(2t + \frac{1}{t}\right) dt = [t^2 + \ln t]_1^e = e^2 \text{ since } \ln e = 1, \ln 1 = 0.$$

### Question #9.

For  $\mathbf{r}(t) = (t \sin t, t \cos t, 2t)$

$$\mathbf{r}'(t) = (\sin t + t \cos t, \cos t - t \sin t, 2) \quad (\text{product rule})$$

for the origin  $t = 0$

**for a)** so  $\mathbf{d} = \mathbf{r}'(0) = (0, 1, 2)$

and an equation of the tangent is  $(x, y, z) = t(0, 1, 2)$  or  $x = 0, z = 2y$

**For b)**

for arclength we need  $\|\mathbf{r}'(t)\|$

$$\|\mathbf{r}'(t)\|^2 = (\sin t + t \cos t)^2 + (\cos t - t \sin t)^2 + 4 = \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t + \cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t + 4 = 5 + t^2 \text{ since } \sin^2 t + \cos^2 t = 1$$

now ,

$t = 0$  for the origin and  $t = \frac{\pi}{2}$  for the point  $A\left(\frac{\pi}{2}, 0, \pi\right)$

$$\begin{aligned} \text{so arclength } s &= \int_0^{\frac{\pi}{2}} \|\mathbf{r}'(t)\| dt = \int_0^{\frac{\pi}{2}} \sqrt{5 + t^2} dt = (\text{Table } a = \sqrt{5}) \\ &= \left[ \frac{t}{2} \sqrt{5 + t^2} + \frac{5}{2} \ln \left( t + \sqrt{5 + t^2} \right) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \sqrt{5 + \frac{\pi^2}{4}} + \frac{5}{2} \ln \left( \frac{\pi}{2} + \sqrt{5 + \frac{\pi^2}{4}} \right) - \frac{5}{2} \ln \sqrt{5}. \end{aligned}$$

**Question #10.**

Find a parametrization of the curve  $c$  given as the intersection of two surfaces

$$c = \{x^2 + y^2 = 2z\} \cap \{3x - 4y - z = 0\}.$$

from the plane  $z = 3x - 4y$  into the paraboloid  $x^2 + y^2 = 2(3x - 4y)$

$$x^2 - 6x + y^2 + 8y = 0 \quad (x - 3)^2 + (y + 4)^2 = 25$$

so  $\left(\frac{x-3}{5}\right)^2 + \left(\frac{y+4}{5}\right)^2 = 1$  thus a parametrization

$$x = 3 + 5 \cos t \quad y = -4 + 5 \sin t \quad z = 25 + 15 \cos t - 20 \sin t, t \in [0, 2\pi).$$