MATH 349 Solutions To Quiz # 2 (Thursday)

1. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$

Obviously the series is of positive terms.

Here $a_n = \dfrac{3^n (n!)^2}{(2n)!}$,

$$a_{n+1} = \frac{3^{n+1}[(n+1)!]^2}{(2n+2)!} = \frac{3.\,3^n[(n+1)n!]^2}{(2n+2)(2n+1)(2n)!} = \frac{3.\,3^n(n+1)^2(n!)^2}{2(n+1)(2n+1)(2n)!} = \frac{3.\,3^n(n+1)(n!)^2}{2(2n+1)(2n)!}$$

Therefore , $\dfrac{a_{n+1}}{a_n} = \dfrac{3.\,3^n(n+1)(n!)^2}{2(2n+1)(2n)!} \cdot \dfrac{(2n)!}{3^n(n!)^2} = \dfrac{3(n+1)}{2(2n+1)}$

Now $L = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1)}{2(2n+1)} = \frac{3}{2} \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Note : You may use L' Hôpital's Rule to Compute the limit above !

Since $L = \dfrac{3}{4} < 1$, the series **Converges**.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ Here $a_n = \dfrac{e^{-\sqrt{n}}}{\sqrt{n}}$, $n \geq 1$

Let $a_n = f(n)$, hence $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$, $x \geq 1$.

Clearly $f(x)$ is **Positive** , and is **Continuous**. Further more $f(x)$ is strictly **decreasing** on the interval $[1\ ,\ \infty)$. This is because

$$f'(x) = \frac{\sqrt{x}\, e^{-\sqrt{x}}\left(-\dfrac{1}{2\sqrt{x}}\right) - e^{-\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x})^2} = -\frac{e^{-\sqrt{x}}\left(1 + \dfrac{1}{\sqrt{x}}\right)}{2\,x} < 0 \text{ , for } x \geq 1.$$

It follow That All Three Conditions of the Integral Test are Satisfied.

Now , consider the Improper Integral $J = \displaystyle\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

$$J = \lim_{R\to\infty} \int_1^{R} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \qquad (1)$$

Consider the indefinite integral $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

Using the Substitution $u = -\sqrt{x}$, we have $du = -\dfrac{1}{2\sqrt{x}}\, dx$, or $\dfrac{1}{\sqrt{x}}\, dx = -2\, du.$

It follows that $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx. = \int e^{-\sqrt{x}}\left(\frac{1}{\sqrt{x}}\, dx\right) = -2\int e^{u}\, du = -2\, e^{u} = -2\, e^{-\sqrt{x}}$

Note : No need for arbitrary constant since integral J is definite.

Therefore (1) becomes

$$J = -2 \lim_{R \to \infty} [\, e^{-\sqrt{x}}\ ]_{x=1}^{x=R} = -2 \lim_{R \to \infty} (\, e^{-\sqrt{R}} - e^{-1}) = -2\,(0 - \tfrac{1}{e}) = \tfrac{2}{e} < \infty$$

Hence the improper integral J **converges and so does the given series**.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n + \ln(n)}$

Note first that $\dfrac{\ln(n)}{n + \ln(n)} > 0$, for $n \geq 2.$

Let $a_n = \dfrac{\ln(n)}{n + \ln(n)}.$

For $n \geq 3,$ we have

$$\ln(n) > 1 \qquad\qquad (*)$$

On the other hand , we have

$$\ln(n) < n$$

Adding n to each side , we have

$$n + \ln(n) < n + n = 2n$$

Hence $$\frac{1}{n + \ln(n)} > \frac{1}{2n} \qquad\qquad (**)$$

It follows from $(*)$, and $(**)$ that

$$a_n = \frac{\ln(n)}{n + \ln(n)} > \frac{1}{2n} = b_n \ , \quad n \geq 3.$$

But $\sum b_n = \frac{1}{2}\sum \frac{1}{n}$ is a $P - Series$ with $P = 1$, hence it **Diverges**.

Note : The constant Multiple does not affect **convergence or divergence** of the series.

Therefore by **Comparison Test** , the given series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n + \ln(n)}$ **Diverges** as well.

2. $\displaystyle\sum_{n=3}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

Let $a_n = \dfrac{\sin^2(n)}{n\sqrt{n}}$, $n \geq 3.$

Obviously $0 < \sin^2(n) < 1$, for $n \geq 3$.

It follows that $a_n = \dfrac{\sin^2(n)}{n\sqrt{n}} < \dfrac{1}{n\sqrt{n}} = b_n$, $n \geq 3$.

But $\sum \dfrac{1}{n\sqrt{n}} = \sum \dfrac{1}{n^{3/2}}$ is a $P - Series$ with $P = \dfrac{3}{2} > 1$, hence it **Converges**.

Therefore by the **Comparison Test** the given series $\sum_{n=3}^{\infty} \dfrac{\sin^2(n)}{n\sqrt{n}}$ **Converges** as well.

3. $\sum_{n=1}^{\infty} \dfrac{3^{n+1}}{8^{\frac{2n}{3}}}$.

Note first that $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4,$ hence $8^{\frac{2n}{3}} = \left(8^{\frac{2}{3}}\right)^n = (4)^n = 4^n$

Series Becomes $\sum_{n=1}^{\infty} \dfrac{3^{n+1}}{4^n}$

To find the sum **S** , we have few options :

<u>**Option 1**</u> :

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n} = \sum_{n=1}^{\infty} \frac{3.3^n}{4^n} = 3\sum_{n=1}^{\infty} \frac{3^n}{4^n} = 3\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Obviously the series is Geometric with common ratio

$$r = \frac{3}{4} < 1 \quad \text{(it Converges!)}$$

Using the sum formula : $a\sum_{n=n_0}^{\infty} r^n = a\,\dfrac{r^{n_0}}{1-r}$ with $a = 3$, $r = \dfrac{3}{4}$, and $n_0 = 1$, we have

$$\mathbf{S} = 3.\,\frac{\left(\frac{3}{4}\right)^1}{1-\frac{3}{4}} = 3.\,\frac{\frac{3}{4}}{\frac{1}{4}} = 3.\,\frac{3}{4}.\frac{4}{1} = 9$$

<u>**Option 2**</u> :

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n} = \frac{9}{4} + \frac{27}{16} + \frac{81}{64} + \ldots$$

Obviously the series is Geometric with first term $a = \dfrac{9}{4}$, and common ratio

$$r = \frac{a_2}{a_1} = \frac{\frac{27}{16}}{\frac{9}{4}} = \frac{27}{16}.\frac{4}{9} = \frac{3}{4} < 1 \,,$$

hence series converges and its sum **S** is given by

$$\mathbf{S} = \frac{a}{1-r} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{\frac{9}{4}}{\frac{1}{4}} = \frac{9}{4}.\frac{4}{1} = 9.$$

END