MATH 349

Solutions To Quiz # 2 (Tuesday)

1. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{e^{2n} n!}$

Obviously the series is of positive terms.

Here $a_n = \dfrac{n^n}{e^{2n} n!}$, $a_{n+1} = \dfrac{(n+1)^{n+1}}{e^{2(n+1)}(n+1)!} = \dfrac{(n+1)(n+1)^n}{e^2 e^{2n}(n+1)n!} = \dfrac{(n+1)^n}{e^2 e^{2n} n!}$

Therefore , $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^n}{e^2 e^{2n} n!} . \dfrac{e^{2n} n!}{n^n} = \dfrac{1}{e^2} \dfrac{(n+1)^n}{n^n} = \dfrac{1}{e^2}\left(\dfrac{n+1}{n}\right)^n = \dfrac{1}{e^2}\left(1+\dfrac{1}{n}\right)^n$

Now $L = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{e^2}\left(1+\frac{1}{n}\right)^n = \frac{1}{e^2} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{e^2} . \; e = \frac{1}{e}$

Note : $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ or you may use L' Hôpital's Rule to Compute it.

Since $L = \dfrac{1}{e} < 1$, the series **Converges**.

(b) $\displaystyle\sum_{n=1}^{\infty} n\, e^{-n^2}$

Here $a_n = n\, e^{-n^2}$, $n \ge 1$

Let $a_n = f(n)$, hence $f(x) = x\, e^{-x^2}$, $x \ge 1.$

Clearly $f(x)$ is **Positive** , and is **Continuous**. Further more $f(x)$

is strictly **decreasing** on the interval $[1 , \infty)$. This is because

$f'(x) = 1. e^{-x^2} + x\, e^{-x^2}(-2x) = e^{-x^2}(1-2x^2) < 0$, for $x \ge 1.$

It follow That All Three Conditions of the Integral Test are Satisfied.

Now , consider the Improper Integral $J = \displaystyle\int_1^{\infty} f(x)\, dx = \int_1^{\infty} x\, e^{-x^2}\, dx.$

$$J = \lim_{R\to\infty} \int_1^R x\, e^{-x^2}\, dx \qquad (1)$$

Consider the indefinite integral $\displaystyle\int x\, e^{-x^2}\, dx.$

Using the Substitution $u = -x^2$, we have $du = -2x\, dx$, or $x\, dx = -\dfrac{1}{2}\, du.$

It follows that $\int x\, e^{-x^2}\, dx. = -\frac{1}{2}\int e^u\, du = -\frac{1}{2}\, e^u = -\frac{1}{2} e^{-x^2}$

Note : No need for arbitrary constant since integral J is definite.

Therefore (1) becomes

$$J = -\frac{1}{2} \lim_{R \to \infty} \left[e^{-x^2} \right]_{x=1}^{x=R} = -\frac{1}{2} \lim_{R \to \infty} \left(e^{-R^2} - e^{-1} \right) = -\frac{1}{2} \left(0 - \frac{1}{e}\right) = \frac{1}{2e} < \infty$$

Hence the improper integral J **converges and so does the given series**.

(c) $\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$

Note first that the series is of positive terms since $0 < \frac{1}{n} < 1 < \frac{\pi}{2}$.

Hence $\frac{1}{n}$ belongs to the first Quadrant where the **sine** function is positive

Let $a_n = \sin^3\left(\frac{1}{n}\right)$.

Recall the Inequality : $\sin(\theta) < \theta$ for all $\theta > 0$.

It follows that $a_n = \sin^3\left(\frac{1}{n}\right) = [\sin\left(\frac{1}{n}\right)]^3 < \left(\frac{1}{n}\right)^3 = \frac{1}{n^3} = b_n$

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a $P-Series$ with $P = 3 > 1$, hence it **Converges** ,

and so does $\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$.

2. $\sum_{n=3}^{\infty} \frac{1}{\ln(\sqrt{n}\,)}$

First observe that $\ln(\sqrt{n}\,) = \ln(n^{1/2}) = \frac{1}{2}\ln(n)$. The series becomes

$$\sum_{n=3}^{\infty} \frac{2}{\ln n} = 2 \sum_{n=3}^{\infty} \frac{1}{\ln n}$$

Note : The constant Multiple does not affect **convergence or divergence** of the series.

Hence you may consider the series $\sum_{n=3}^{\infty} \frac{1}{\ln n}$.

Let $a_n = \frac{1}{\ln(n)}$, $n \geq 3$.

Recall the Inequality $0 < \ln(n) < n$, for $n \geq 2$.

It follows that $a_n = \frac{1}{\ln(n)} > \frac{1}{n} = b_n \quad , \; n \geq 3.$

But $\sum \frac{1}{n}$ is the Well Known Harmonic Series , it **Diverges**.

Therefore by **Comparison Test** the series $\sum_{n=3}^{\infty} \frac{1}{\ln(\sqrt{n}\,)}$ **Diverges** as well.

3. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{9^{\frac{n}{2}}}.$

Note first that $9 = 3^2$. Hence $9^{\frac{n}{2}} = (3^2)^{n/2} = 3^n$

Series Becomes $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n}$

To find the sum **S** , we have few options :

Option 1 : $$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{2.2^n}{3^n} = 2 \sum_{n=1}^{\infty} \frac{2^n}{3^n} = 2 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

Obviously the series is Geometric with common ratio

$$r = \frac{2}{3} < 1$$

Using the sum formula : $a \sum_{n=n_0}^{\infty} r^n = a \frac{r^{n_0}}{1-r}$ with $a = 2$, $r = \frac{2}{3}$, and $n_0 = 1$, we have

$$\mathbf{S} = 2. \frac{\left(\frac{2}{3}\right)^1}{1 - \frac{2}{3}} = 2. \frac{\frac{2}{3}}{\frac{1}{3}} = 2. \frac{2}{3} . \frac{3}{1} = 4$$

Option 2 : $$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$$

Obviously the series is Geometric with first term $a = \frac{4}{3}$, and common ratio

$$r = \frac{a_2}{a_1} = \frac{\frac{8}{9}}{\frac{4}{3}} = \frac{8}{9} . \frac{3}{4} = \frac{2}{3} < 1 \quad ,$$

hence series converges and its sum **S** is given by

$$\mathbf{S} = \frac{a}{1-r} = \frac{\frac{4}{3}}{1 - \frac{2}{3}} = \frac{\frac{4}{3}}{\frac{1}{3}} = \frac{4}{3} . \frac{3}{1} = 4$$

END