Solutions To Quiz (5) –Tuesday

1. Let $f(x,y,z) = \sqrt{z} + \sin(xy)$, and let $h(t) = f(x,y,z)$, where $x(t) = t^2,\ y(t) = \ln(t)$, and $z(t) = 4\,e^{t-1}$

   Use the **Chain Rule** to find $\frac{dh}{dt}$ at $t = 1$. (**3 Marks**)

Solution :

By the chain rule : $\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$ (1)

Here $f(x,y,z) = \sqrt{z} + \sin(xy)$

Therefore, $\begin{cases} \frac{\partial f}{\partial x} = y\cos(xy) \\ \frac{\partial f}{\partial y} = x\cos(xy) \\ \frac{\partial f}{\partial z} = \frac{1}{2\sqrt{z}} \end{cases}$ (2)

Next : $\begin{cases} \frac{dx}{dt} = 2t \\ \frac{dy}{dt} = \frac{1}{t} \\ \frac{dz}{dt} = 4\,e^{t-1} \end{cases}$ (3)

Substituting (2) & (3) into (1) , we have

$$\frac{dh}{dt} = (y\cos(xy))(\,2t) + (x\cos(xy))\,(\frac{1}{t}) + (\frac{1}{2\sqrt{z}})(4\,e^{t-1})$$

Note that at $t = 1$, we have

$$x = (1)^2 = 1\ ,\ \ y = \ln(1) = 0\ ,\ and\ \ z = 4\,e^{1-1} = 4\,e^0 = 4$$

It follows that

$$\left.\frac{dh}{dt}\right|_{t=1} = \left.(y\cos(xy))(2t) + (x\cos(xy))(\frac{1}{t}) + (\frac{1}{2\sqrt{z}})(4\,e^{t-1})\right|_{t=1\ ,\ (x,y,z)=(1,0,4)}$$

$$= (0)\,(2) + (1)(\cos(0))(\frac{1}{1}) + (\frac{1}{2\sqrt{4}})(4e^0) = 0 + 1 + 1 = 2$$

2. Find an equation of the plane tangent to the surface $z = \frac{x}{y} - \frac{4y}{x} + 1$, at the point on the surface

   where $(x,y) = (2,1)$. (**3 Marks**)

Solution :

First at $x = 2\ ,\ y = 1\ ,$ we have $z = \frac{2}{1} - \frac{4(1)}{2} + 1 = 2 - 2 + 1 = 1$

Therefore a point on the surface is given by $P\,(x_0,y_0,z_0) = (2\ ,\ 1\ ,\ 1)$

A vector normal to tangent plane at $P\,(2,1,1)$ is given by $\left.\overrightarrow{N} = (\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1)\right|_{p\,(2,1,1)}$

Here , $f(x,y) = \frac{x}{y} - \frac{4y}{x} + 1 = xy^{-1} - 4x^{-1}y + 1$, hence

$$f_x(x,y) = y^{-1} + 4x^{-2}y = \frac{1}{y} + \frac{4y}{x^2} \quad , \quad \text{and } f_y(x,y) = -xy^{-2} - 4x^{-1} = -\frac{x}{y^2} - \frac{4}{x}.$$

It follows that

$$A = f_x(2,1) = \frac{1}{1} + \frac{4(1)}{2^2} = 1 + 1 = 2 \text{ , and}$$

$$B = f_y(2,1) = -\frac{2}{(1)^2} - \frac{4}{2} = -2 - 2 = -4$$

Hence a vector normal to tangent plane at P is given by

$$\overrightarrow{N} = (A,B,-1) = (2\,,-4\,,-1)$$

Equation of tangent plane is thus given by

$$z = z_0 + A(x - x_0) + B(y - y_0)$$

$$\Rightarrow \quad z = 1 + 2(x-2) + (-4)(y-1)$$

$$\Rightarrow \quad z = 1 + 2(x-2) - 4(y-1)$$

$$\Rightarrow \quad z = 1 + 2x - 4 - 4y + 4$$

$$\Rightarrow \quad z = 2x - 4y + 1$$

$$or \quad 2x - 4y - z + 1 = 0$$

3. Let $f(x,y) = 2x^2 + 2xy - y^2$ and let P be the point $(1,-1)$.

(a) In what directions at the point P does the function f has the rate of change equal to 2 ? (**2 Marks**)

Solution :

Recall : The Directional Derivative of f at the point (x_0, y_0) in the direction of the

unit vector $\overrightarrow{v} = (a\,,b)$ is given by

$$D_v f(x_0,y_0) = \overrightarrow{\nabla} f(x_0,y_0) \cdot \overrightarrow{v}$$

Therefore $\quad D_v f(2\,,-2) = \overrightarrow{\nabla} f(1\,,-1) \cdot (a\,,b)$

$$2 = \overrightarrow{\nabla} f(1\,,-1) \cdot (a\,,b) \qquad (1)$$

Here $f(x,y) = 2x^2 + 2xy - y^2$, hence

$$f_x(1\,,-1) = 4x + 2y|_{(x\,,\,y)=(1\,,-1)}$$

$$= 4(1) + 2(-1)$$

$$= 4 - 2 = 2$$

and

$$f_y(1\,,-1)) = 2x - 2y|_{(x\,,\,y)=(1\,,-1)}$$

$$= 2(1) - 2(-1)$$

$$= 2 + 2 = 4$$

$$\vec{\nabla} f(1\,,-1) = (f_x(1\,,-1)\,, f_y(1\,,-1)) = (2\,,\,4) \qquad (2)$$

Substituting (2) into (1) , we have

$$2 = (2\,,\,4) \cdot (a\,,b)$$

$$\Rightarrow \quad 2 = 2a + 4b$$

$$\Rightarrow \quad a + 2b = 1$$

$$\Rightarrow \quad a = 1 - 2b$$

Since $\vec{v} = (a\,,b)$ is a **Unit Vector** , we have

$$a^2 + b^2 = 1$$

Substituting $a = 1 - 2b$ into equation above , we have

$$(1 - 2b)^2 + b^2 = 1$$

$$\Rightarrow \quad 1 - 4b + 4b^2 + b^2 = 1$$

$$\Rightarrow \quad 5b^2 - 4b = 0$$

$$\Rightarrow \quad b(5b - 4) = 0$$

$$\Rightarrow \quad \textit{either } b = 0 \textit{ or } b = \frac{4}{5}.$$

Let us find the corresponding values of a :

If $b = 0$, then $a = 1 - 2(0) = 1$, hence $\vec{v_1} = (a\,,b) = (1,0)$

and if $b = \frac{4}{5}$, then $a = 1 - 2\left(\frac{4}{5}\right) = 1 - \frac{8}{5} = -\frac{3}{5}$, hence $\vec{v_2} = (a\,,b) = (-\frac{3}{5}\,,\,\frac{4}{5})$

Therefore there are two possible directions , namely $\vec{v_1} = (1,0)$ & $\vec{v_2} = (-\frac{3}{5}\,,\,\frac{4}{5})$

(b) In what direction from the point P does this function f decreases most rapidly and what is this most rapid rate of decrease ? (**2Marks**)

The function f decreases most rapidly in the direction of the unit vector $\vec{n} = -\dfrac{\vec{\nabla} f(1,-1)}{\left\| \vec{\nabla} f(1,-1) \right\|}$

Hence from (2) , we have

$$\vec{n} = -\frac{(2,4)}{\|(2,4)\|} = -\frac{(2,4)}{\sqrt{2^2+4^2}} = -\frac{(2,4)}{\sqrt{20}} = -\frac{1}{2\sqrt{5}}\,(2,4)$$

$$\Rightarrow \qquad \vec{n} = \left(\frac{-1}{\sqrt{5}}\,,\,\frac{-2}{\sqrt{5}} \right)$$

The most rapid rate of decrease (Minimum Rate) is given by $-\left\| \vec{\nabla} f(1,-1) \right\| = -\|(2,4)\| = -2\sqrt{5}$ as shown above!