MATH349 – Assignment 4

Due: Wednesday June 15

1. Express the following points, which are given in Cartesian coordinates, in polar coordinates:
$$P = (1,1)\ , \quad Q = (-3,4)\ , \quad R = (0,-7)\ .$$

2. Find the arclength of the following curves
   (a) $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$ between $t = 0$ and $t = 2\pi$.
   (b) $X(t) = (e^{3t}, e^{-3t}, 3\sqrt{2}t)$ between $t = 0$ and $t = \frac{1}{3}$.

3. The Folium is defined as the set of points in $\mathbb{R}^2$ which satisfy the equation
$$x^3 + y^3 = 3axy$$
where a is a fixed constant.
   (a) Roughly, draw the Folium in the case that $a = 1$.
   (b) Draw a line of slope t joining $(0,0)$ to another point P on the Folium.
   (c) Find rational functions $x = u(t)$ and $y = v(t)$ which parametrize the Folium by finding the second point of intersection of the line $y = tx$ through the origin.

4. The Roses of Grandi are defined by the polar equation
$$r = a\cos n\theta$$
for integer values of n and a is a fixed constant.
   (a) Show that the "rose" for $n = 1$ is a circle.
   (b) Show that the "rose" for $n = 2$ satisfies the equation
$$(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$$
   (c) Roughly, plot the set of points which satisfy the equation in (b).

5. Verify that $\mathbf{r}(t) = \mathbf{r}_0\cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)$ satisfies the initial value problem
$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}\ , \quad \mathbf{r}'(0) = \mathbf{v}_0\ , \quad \mathbf{r}(0) = \mathbf{r}_0\ .$$
(It is the unique solution.) Describe the path $\mathbf{r}(t)$. What is the path if $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$?

6. Let $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$ and let $\theta(t)$ be the angle which the tangent line at a given point of the curve makes with the z-axis. Show that $\theta(t)$ is the constant $\dfrac{b}{\sqrt{a^2 + b^2}}$.

7. Show that if $\kappa(s) = c$ is a positive constant and $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circle. (Note that here the curve is parametrized in terms of arc length.)

8. Find the radius of curvature of the curve
$$\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$
at the point $t = 1$.

Bonus question: (This question is about Kepler's Laws which can be found in section 11.6 of Adams.) Show that the orbital speed of a planet is constant if and only if the orbit is circular. (Hint: Use the conservation of energy identity.)