MATH349 – Assignment 1 – Solutions

[3 marks] 1. Write the first five terms of each of the following sequences:

(a) $\left\{\dfrac{3+(-1)^n \sin(\pi n/2)}{n!}\right\}$.

(b) $\{a_n\}$ where $a_n = \dfrac{1}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

(c) $\{x_n\}$ where $x_n = \dfrac{1}{(2n) \cdot x_{n-1}}$ and $x_1 = 2$.

The first five terms are:

$$\left\{2, \frac{3}{2}, \frac{2}{3}, \frac{1}{8}, \frac{1}{60}\right\},$$

$$\left\{\frac{1}{2}, \frac{1}{8}, \frac{1}{48}, \frac{1}{384}, \frac{1}{3840}\right\},$$

and

$$\left\{2, \frac{1}{8}, \frac{4}{3}, \frac{3}{32}, \frac{16}{15}\right\}$$

respectively.

[1 mark] 2.(a) Give an example of a sequence which is bounded but is not convergent.

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

[1 mark] (b) Give an example of a sequence which is increasing and converges to 0.

$$\{x_n\} \text{ where } x_n = \frac{-1}{n}$$

This sequence is increasing because

$$\frac{-1}{n+1} > \frac{-1}{n} \quad \text{which implies} \quad x_{n+1} > x_n.$$

[2 marks] 3. Find sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n - y_n$ converges to 0, $\sum x_n$ diverges and $\sum y_n$ converges.

Take $y_n = 0$ and $x_n = \dfrac{1}{n}$.

[4 marks] 4. Show that the sequence $\{u_n\}$ where $u_n = \dfrac{\sqrt{n}}{n+1}$ is

(a) monotonically decreasing
(b) bounded above
(c) bounded below
(d) has a limit

(a) Since $u_n > 0$, the statement $u_n \geqslant u_{n+1}$ is equivalent to the statement $u_n^2 \geqslant u_{n+1}^2$. We have

$$u_{n+1}^2 = \frac{n+1}{(n+2)^2} \quad \text{and} \quad u_n^2 = \frac{n}{(n+1)^2}.$$

Then,

$$\begin{aligned}(n+1)^2(n+2)^2\left(u_n^2 - u_{n+1}^2\right) &= n(n+2)^2 - (n+1)^3 \\ &= n^3 + 4n^2 + 4n - (n^3 + 3n^2 + 3n + 1) \\ &= n^2 + n - 1 > 0 \quad \text{for all } n > 0.\end{aligned}$$

Since $(n+1)^2(n+2)^2 > 0$ for all n, we must have

$$u_n^2 - u_{n+1}^2 > 0 \quad \text{which implies} \quad u_n^2 > u_{n+1}^2,$$

that is, $\{u_n\}$ is monotonically decreasing.

(b) By (a), u_n is monotonically decreasing, thus $\{u_n\}$ is bounded above by u_1 which is equal to 1/2.

(c) Since $\sqrt{n} > 0$ and $n+1 > 0$ for all $n > 0$, we have $u_n > 0$ for all $n > 0$. That is, $\{u_n\}$ is bounded below by 0.

(d) By (b) and (c), $\{u_n\}$ is bounded above by 1/2 and bounded below by 0. Therefore it is bounded.

(e) By (a) and (d), $\{u_n\}$ is bounded and monotonically decreasing. Thus it satisfies the conditions of the monotonic sequence test, hence it is convergent and has a limit.

[4 marks] 5. Consider the series $\displaystyle\sum_{i=2}^{\infty} \left(\frac{2}{3}\right)^i$.

(a) Write the first four partial sums of this series.
(b) Series of this type were given a name in class. What is it?
(c) Show that this series is convergent.
(d) Find the value of the series.

(Warning: Note the series starts at $i = 2$.)

(a) The first four partial sums are:

$$\begin{aligned}S_1 &= \left(\frac{2}{3}\right)^2 = \frac{4}{9} \\ S_2 &= \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 = \frac{4}{9} + \frac{8}{27} = \frac{20}{27} \\ S_3 &= \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 = \frac{4}{9} + \frac{8}{27} + \frac{16}{81} = \frac{76}{81} \\ S_4 &= \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 = \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} = \frac{260}{243}\end{aligned}$$

(b) Series of this type were called geometric series.

(c) The common ratio for this geometric series is 2/3. Since this is less than 1, the series is convergent.

(d) Since the series is convergent, we may write

$$\begin{aligned}\sum_{i=2}^{\infty}\left(\frac{2}{3}\right)^i &= \frac{4}{9}\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i \\ &= \frac{4}{9}\cdot\frac{1}{1-\frac{2}{3}} \\ &= \frac{4}{9}\cdot\frac{1}{\frac{1}{3}} = \frac{4}{3}\end{aligned}$$

[3 marks] 6. Show that $\sum_{n=1}^{\infty}\frac{4}{(4n-1)(4n+3)} = \frac{1}{3}$.

First, we use partial fractions; that is, find A and B such that

$$\frac{4}{(4n-1)(4n+3)} = \frac{A}{4n-1} + \frac{B}{4n+3}$$

Clearing denominators gives

$$4 = A(4n+3) + B(4n-1)$$

which is equivalent to

$$4 = 4n(A+B) + (3A-B) \qquad (\dagger)$$

Since (†) is true for all n, we must have

$$4 = (3A-B) \quad \text{and} \quad (A+B) = 0.$$

These two equations yield the unique solution

$$A = 1 \quad \text{and} \quad B = -1.$$

Thus,

$$\frac{4}{(4n-1)(4n+3)} = \frac{1}{4n-1} - \frac{1}{4n+3}.$$

Hence the k-th partial sum, S_k, of the series is given by

$$\begin{aligned}S_k &= \sum_{n=1}^{k}\frac{4}{(4n-1)(4n+3)} \\ &= \sum_{n=1}^{k}\left(\frac{1}{4n-1} - \frac{1}{4n+3}\right) \\ &= \left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{11}\right) + \left(\frac{1}{11}-\frac{1}{15}\right) + \cdots + \left(\frac{1}{4k-1}-\frac{1}{4k+3}\right) \\ &= \frac{1}{3} - \frac{1}{4k+3}\end{aligned}$$

As $k \to \infty$, we have

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{1}{3} - \frac{1}{4k+3}\right) = \frac{1}{3}.$$

Therefore the series is convergent with value 1/3, i.e.

$$\sum_{n=1}^{\infty} \frac{4}{(4n-1)(4n+3)} = \frac{1}{3}$$

[1 mark] 7.(a) State the Ratio Test.

Suppose that a_n is ultimately positive and that

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \text{(including } L = \infty\text{)}$$

Then

- If $0 \le L < 1$, then $\sum a_n$ converges.
- If $L > 1$, then $\sum a_n$ diverges.

[2 marks] (b) Use the Ratio Test to determine whether $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ is convergent.

Let $a_n = \frac{3^n}{n^3}$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n} \\ &= 3\left(\frac{n}{n+1}\right)^3 \\ &= 3\left(\frac{1}{1+1/n}\right)^3\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3$. Hence, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ is divergent.

[1 mark] 8.(a) State the Integral Test.

If f is a positive, continuous and decreasing function on (N, ∞) for some $N > 0$ and $f(n) = a_n$ then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_N^{\infty} f(t)dt$$

either both converge or both diverge.

[2 marks] (b) Use the Integral Test to determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent.

Let $f(x) = xe^{-x^2}$. Then f is positive on $(1, \infty)$ since e^{-x^2} is positive for all x. Also, f is continuous since x and e^{-x^2} are continuous. Lastly,

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$$

so $f'(x) < 0$ on $(1, \infty)$. Therefore f is decreasing on $(1, \infty)$. Hence f satisfies the conditions of the Integral test. Next,

$$\begin{aligned}\lim_{m\to\infty} \int_1^m te^{-t^2} dt &= \lim_{m\to\infty} \left[-\tfrac{1}{2}e^{-t^2}\right]_1^m \\ &= \lim_{m\to\infty} \left(\tfrac{1}{2}e^{-1} - \tfrac{1}{2}e^{-m^2}\right) \\ &= \tfrac{1}{2}e^{-1}\end{aligned}$$

Thus the integral $\displaystyle\int_1^\infty f(t)dt$ exists, so by the Integral Test, the series $\displaystyle\sum_{n=1}^\infty ne^{-n^2}$ is convergent.

[1 mark] 9. Explain why the following argument is false.

If $S = 1-1+1-1+1-1+\cdots$, then $S = 1-(1-1)-(1-1)-(1-1)-\cdots = 1$. Also, $S = (1-1)+(1-1)+(1-1)+\cdots = 0$. Hence $1 = 0$.

The k-th partial sum, S_k, of $1-1+1-1+1-\ldots$ is given by

$$S_k = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

Hence the sequence of partial sums is $\{1, 0, 1, 0, 1, 0, \ldots\}$. Since this sequence does not converge, the sum $S = 1-1+1-1+\ldots$ does not converge. Thus, S is not equal to either 0 or to 1. Also, observe that if S did converge then the limit must be unique.

[3 marks] Bonus question: Show that the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{4x_n + 2}{x_n + 3} \quad \text{and} \quad x_1 = 3$$

converges to 2.

First note that $x_n > 0$ for all n and $x_2 < x_1$. Next, suppose that $x_n < x_{n-1}$ then

$$x_{n+1} - x_n = \frac{4x_n + 2}{x_n + 3} - \frac{4x_{n-1} + 2}{x_{n-1} + 3} = \frac{10(x_n - x_{n-1}}{(x_n + 3)(x_{n-1} + 3)}$$

Thus $x_{n+1} - x_n < 0$ also. Hence by induction $x_{n+1} < x_n$ for all n. That is, $\{x_n\}$ is a decreasing sequence. Also note that it is bounded below by 0 and bounded above by 3. Thus $\{x_n\}$ satisfies the conditions of the monotonic sequence test and therefore it is convergent.

Now, suppose $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty} x_{n+1} = x$ also. Hence,

$$\begin{aligned}x = \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{4x_n + 2}{x_n + 3} \\ &= \frac{\lim_{n\to\infty} 4x_n + 2}{\lim_{n\to\infty} x_n + 3} \\ &= \frac{4x + 2}{x + 3}\end{aligned}$$

This is equivalent to

$$x^2 - x - 2 = 0$$

which has solutions $x = 2$ and $x = -1$. Since $x_n > 0$, $x \neq -1$, thus $x = 2$, that is

$$\lim_{n \to \infty} x_n = 2.$$
