MATH349 – Assignment 2 – Solutions

[3 marks] 1. Determine the centre, radius of convergence and interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}$$

The centre is 1. The n-th coefficient is $a_n = \dfrac{n}{2^n(3n-1)}$, so

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{n+1}{2^{n+1}(3n+2)} \frac{2^n(3n-1)}{n} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n} \cdot \frac{3n-1}{3n+2} \cdot \frac{1}{2} \right) \\ &= \frac{1}{2}\end{aligned}$$

Therefore the radius of convergence is 2. The endpoints of the interval of convergence are -1 and 3. At $x=-1$ we have

$$\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n(3n-1)} = \sum_{n=1}^{\infty} \frac{n(-1)^n}{(3n-1)}$$

Since the terms of this sequence do not tend to 0 as n tends to infinity, the series does not converge. At $x=3$, we have

$$\sum_{n=1}^{\infty} \frac{n(2)^n}{2^n(3n-1)} = \sum_{n=1}^{\infty} \frac{n}{(3n-1)}$$

Again, this series is divergent. Therefore the interval of convergence is $(-1,3)$.

[3 marks] 2. Consider the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{\log n}$.

(a) Write down the centre and the first five coefficients.
(b) Determine the radius of convergence.
(c) Examine the two endpoints of the interval of convergence.

(a) The centre is -2 and the first five coefficients are:

$$\frac{1}{\log 2}, \frac{1}{\log 3}, \frac{1}{\log 4}, \frac{1}{\log 5}, \frac{1}{\log 6}.$$

(b) The n-th coefficient is $a_n = 1/\log n$. Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\log(n+1)}{\log n} = 1$$

Therefore the radius of convergence is 1.
(c) The two endpoints of the interval of convergence are -3 and -1. At $x=-3$ we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$$

which is convergent by the alternating series test. At $x = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{\log n}$$

which we know is divergent (otherwise we can use the comparison test with the harmonic series.) Thus, the interval of convergence is $[-3, 1)$.

[4 marks] 3. (a) Give the Taylor series for $\sin x$ and $\cos x$ about the point 0.
(b) Find the power series expansion for $\sin x \cos x$ about the point 0. (Use (a).)
(c) Find the Taylor series expansion for $\sin 2x$ about the point 0.
(d) Confirm that $2 \sin x \cos x = \sin 2x$.

(a) The Taylor series about 0 for $\sin x$ is

$$\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Taylor series about 0 for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

(b) Multiplying the above two power series will give the power series expansion for $\sin x \cos x$. This product is

$$x - \left(\frac{1}{3!} + \frac{1}{2!}\right) x^3 + \left(\frac{1}{5!} + \frac{1}{3!\,2!} + \frac{1}{4!}\right) x^5 - \left(\frac{1}{7!} + \frac{1}{5!\,2!} + \frac{1}{3!\,4!} + \frac{1}{6!}\right) x^7 + \left(\frac{1}{9!} + \frac{1}{7!\,2!} + \frac{1}{5!\,4!} + \frac{1}{3!\,6!} + \frac{1}{8!}\right) x^9 + \ldots$$

(c) From the Taylor series for $\sin x$, we find that the Taylor series about 0 for $\sin 2x$ is

$$\frac{2x}{1!} - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \cdots$$

(d) From (b), the $2n+1$-th coefficient of $\sin x \cos x$ is

$$(-1)^n \left(\frac{1}{(2n+1)!} + \frac{1}{(2n-1)!\,2!} + \frac{1}{(2n-3)!\,4!} + \cdots + \frac{1}{3!\,(2n-2)!} + \frac{1}{(2n)!}\right)$$

and this is equal to

$$\frac{(-1)^n}{(2n+1)!} \left[\binom{2n+1}{0} + \binom{2n+1}{2} + \binom{2n+1}{4} + \binom{2n+1}{6} + \cdots + \binom{2n+1}{2n-2} + \binom{2n+1}{2n}\right].$$

In class we saw that

$$0 = \binom{2n+1}{0} - \binom{2n+1}{1} + \binom{2n+1}{2} - \binom{2n+1}{3} + \cdots + \binom{2n+1}{2n} - \binom{2n+1}{2n+1}$$

and question 8 shows that

$$2^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \binom{2n+1}{3} + \cdots + \binom{2n+1}{2n} + \binom{2n+1}{2n+1}$$

Adding these two equations gives

$$2^{2n} = \binom{2n+1}{0} + \binom{2n+1}{2} + \binom{2n+1}{4} + \binom{2n+1}{6} + \cdots + \binom{2n+1}{2n-2} + \binom{2n+1}{2n}$$

This means we may write the expansion for $\sin x \cos x$ as

$$\frac{x}{1!} - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} + \cdots$$

and so $2 \sin x \cos x = \sin 2x$ for all x since all these power series are valid for every x.

[4 marks] 4. (a) Use the binomial theorem to show that

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \ldots$$

(b) Use (a) and $\sin^{-1} x = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$ to show

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots$$

(c) In what region is this expansion valid?

(d) Conclude that

$$\frac{\pi}{2} = 1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \ldots$$

(a) The binomial theorem says

$$\begin{aligned}(1-y)^{-1/2} &= 1 + \frac{1}{2}y + \frac{1}{2!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)y^2 - \frac{1}{3!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)y^3 + \ldots \\ &= 1 + \frac{1}{2}y + \frac{1 \cdot 3}{2 \cdot 4}y^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}y^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}y^4 + \ldots\end{aligned}$$

which is valid for $|y| < 1$. Now let $y = t^2$ and we get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4}t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}t^6 + \ldots$$

valid in the region $-1 < t < 1$.

(b) We may integrate the series in (a) in its interval of convergence. Doing this termwise and recalling

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x$$

we find

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots$$

(c) (This one is tricky) Since the expansion in (a) is valid for x in $(-1, 1)$, the expansion in (b) is also valid in this region. However, it may now also be valid at the endpoints, namely $x = -1$ and $x = 1$. Firstly, note that the expansion is an odd function, so if it is convergent at $x = 1$ it is also convergent at $x = -1$.

At $x = 1$ the series is

$$1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \ldots$$

It is less than the series

$$1+\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\dots$$

This series may be written as

$$1+\sum_{n=1}^{\infty} a_n$$

where

$$a_n=\frac{1\cdot 3\cdots(2n-1)}{2^n n!}$$

If we use the ratio test, we find the limit of a_{n+1}/a_n is 1. However, we can do the following:

$$\begin{aligned}\frac{a_{n+1}}{a_n}=1-\frac{3}{2(n+1)} &\le \left(1-\frac{1}{n+1}\right)^{3/2}\\ &=\left(\frac{n}{n+1}\right)^{3/2}\\ &=\frac{b_{n+1}}{b_n}\end{aligned}$$

where $b_n=n^{-3/2}$. Thus, $\sum b_n$ is convergent and therefore $\sum a_n$ is also convergent by the comparison test.
Thus the interval of convegence is $[-1,1]$.
(d) By taking $x=1$ we get $\sin^{-1}(1)=\pi/2$ and

$$\frac{\pi}{2}=1+\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\dots$$

[3 marks] 5. Find $\int_0^1 \frac{1-\cos x}{x}\,dx$ correct to 3 decimal places.

The Taylor series about 0 for $1-\cos x$ is

$$\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$$

Therefore the Taylor series about 0 for $(1-\cos x)/x$ is

$$\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\dots$$

This expansion is valid for all x and we find

$$\begin{aligned}\int_0^1 \frac{1-\cos x}{x}\,dx &= \left[\frac{x^2}{4}-\frac{x^4}{96}+\frac{x^6}{4320}-\dots\right]_0^1\\ &=\frac{1}{4}-\frac{1}{96}+\frac{1}{4320}-\dots\end{aligned}$$

Evaluating this we find

$$\int_0^1 \frac{1-\cos x}{x}\,dx \approx 0.239...$$

[3 marks] 6. (a) Give the power series expansion for $\log(1+x)$ about the point 0.
(b) Determine the power series expansions about 0 for $x - \log(1+x)$ and $x\log(1+x)$.
(c) Use (b) to show

$$\lim_{x\to 0}\left(\frac{1}{\log(1+x)} - \frac{1}{x}\right) = \frac{1}{2}$$

(a) The expansion for $\log(1+x)$ is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

(b) From (a) we find

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots$$

and

$$x\log(1+x) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots$$

Both of these expansions are valid in the region $-1 < x \leq 1$.
(c) In the region $-1 < x \leq 1$ we may write

$$\begin{aligned}\frac{1}{\log(1+x)} - \frac{1}{x} = \frac{x - \log(1+x)}{x\log(1+x)} &= \frac{x^2/2 - x^3/3 + x^4/4 - \cdots}{x^2 - x^3/2 + x^4/3 - \cdots}\\ &= \frac{1/2 - x/3 + x^2/4 - \cdots}{1 - x/2 + x^2/3 - \cdots}\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\log(1+x)} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{1/2 - x/3 + x^2/4 - \cdots}{1 - x/2 + x^2/3 - \cdots} = \frac{1}{2}$$

[3 marks] 7. Determine the coefficient of x^4 in the power series expansion about 0 of $\dfrac{1}{(1+x+x^2)^4}$.

We have

$$\begin{aligned}\frac{1}{1+x+x^2} = \frac{1}{1-(-x-x^2)} &= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 + \cdots\\ &= 1 - x - x^2 + (x^2 + 2x^3 + x^4) - (x^3 + 3x^4 + 3x^5 + x^6)\\ &\qquad + (x^4 + 4x^5 + 6x^6 + 4x^7 + x^8) + \cdots\\ &= 1 - x + x^3 - x^4 + \cdots\end{aligned}$$

Since we are interested in the coefficient of x^4, we only need to consider the terms up to x^4. From above, we find

$$\begin{aligned}\frac{1}{(1+x+x^2)^2} &= (1 - x + x^3 - x^4 + \cdots)(1 - x + x^3 - x^4 + \cdots)\\ &= 1 - 2x + x^2 + 2x^3 - 4x^4 + \cdots\end{aligned}$$

Therefore

$$\frac{1}{(1+x+x^2)^4} = (1 - 2x + x^2 + 2x^3 - 4x^4 + \cdots)(1 - 2x + x^2 + 2x^3 - 4x^4 + \cdots)$$

And the coefficient of x^4 is

$$-4 + 2(-2) + 1 + (-2)2 - 4 = -15.$$

(You could also use the binomial theorem to answer this question.)

[2 marks] 8. Use the binomial theorem to show the following

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

The binomial theorem says that

$$(1+x)^n = \binom{n}{0}x^n + \binom{n}{n-1}x^{n-1} + \binom{n}{n-2}x^{n-2} + \cdots + \binom{n}{1}x + \binom{n}{0}.$$

If we take $x = 1$ in the above formula, we find

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

[2 marks] Bonus question: Show that

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3\log 2}{2}$$

First note that the sequence is not absolutely convergent so we may not alter the order of summation. The trick is to notice that $\frac{3}{2} = 1 + \frac{1}{2}$. We know that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} a_n$$

and so

$$\frac{1}{2}\log 2 = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + \cdots = \sum_{m=1}^{\infty} b_m$$

Since both of the series are convergent we may add them to find

$$\frac{3}{2}\log 2 = \sum_{i=1}^{\infty} a_i + b_i = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$$
