MATH349 – Assignment 3 – Solutions

[3 marks] 1. For the points:

$$P = (1, 2, -3)\,, \quad Q = (2, -3, 4)\,, \quad R = (-3, 1, 2)\,.$$

(a) Find the line passing through P and Q.
(b) Give the equation of the plane passing through all three points P, Q, R.

(a) The line passing through P and Q has direction

$$Q - P = (2-1)\mathbf{i} + (-3-2)\mathbf{j} + (4-(-3))\mathbf{k} = \mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$$

It passes through the point P, so the vector parametrizated form of the line is

$$\mathbf{l} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + t(\mathbf{i} - 5\mathbf{j} + 7\mathbf{k})$$

You could also give the line as

$$\begin{aligned} x &= 1 + t \\ y &= 2 - 5t \\ z &= -3 + 7t \end{aligned}$$

(b) If $\mathbf{u} = Q - P$ then $\mathbf{u} = \mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ from (a). The vector joining R and P is given by

$$R - P = (-3-1)\mathbf{i} + (1-2)\mathbf{j} + (2-(-3))\mathbf{k} = -4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

Thus the normal to the plane passing through P, Q, R has direction given by

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & 7 \\ -4 & -1 & 5 \end{pmatrix} = -18\mathbf{i} - 33\mathbf{j} - 21\mathbf{k}$$

Thus the plane has equation $-18x - 33y - 21z = D$ for some D. The plane passes through the point $(1, 2, -3)$, so

$$-18 - 66 + 63 = D \quad \text{which implies} \quad D = -21$$

Therefore the equation of the plane is

$$6x + 11y + 7z = 7.$$

[3 marks] 2. Find all vectors $\mathbf{v}$ in $\mathbb{R}^3$ which satisfy the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

First, if $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ then

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ v_1 & v_2 & v_3 \end{pmatrix} = (2v_3 - 3v_2)\mathbf{i} + (3v_1 + v_3)\mathbf{j} + (-v_2 - 2v_1)\mathbf{k}$$

Thus,

$$\begin{aligned} 2v_3 - 3v_2 &= 1 \\ 3v_1 + v_3 &= 5 \\ -v_2 - 2v_1 &= -3 \end{aligned}$$

This system of linear equations is the same as

$$\begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

Gaussian elimination on this system produces

$$\left(\begin{array}{ccc|c} 0 & -3 & 2 & 1 \\ 6 & 0 & 2 & 10 \\ -6 & -3 & 0 & -9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & -3 & 2 & 1 \\ 6 & 0 & 2 & 10 \\ 0 & -3 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & -3 & 2 & 1 \\ 6 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This system has homogenous solution $(1, -2, -3)$ and a inhomogenous solution of $(0, 3, 5)$. Therefore the set of all solutions is given by

$$v_1 = t\ , \quad v_2 = 3 - 2t\ , \quad v_3 = 5 - 3t.$$

Therefore the set of all vectors $\mathbf{v}$ satisfying the equation are given by the line

$$3\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}).$$

[3 marks] 3. Given the three vectors

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}\,, \quad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\,, \quad \mathbf{w} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\,.$$

(a) Determine $\mathbf{u} + \mathbf{w}$.
(b) Determine $|\mathbf{v}|$.
(c) Find the projection of $\mathbf{u} + \mathbf{w}$ in the direction of $\mathbf{v}$.

(a) $\mathbf{u} + \mathbf{w} = 5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.
(b) $|\mathbf{v}| = \sqrt{1(-2)^2 + 2^2} = \sqrt{9} = 3$.
(c) The projection of $\mathbf{u} + \mathbf{w}$ in the direction of $\mathbf{v}$ is

$$\begin{aligned} \frac{(\mathbf{u}+\mathbf{w}) \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} &= \frac{(5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \bullet (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}{9}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \\ &= \frac{5+6+6}{9}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \frac{17}{9}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \end{aligned}$$

[3 marks] 4. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, is it true that $\mathbf{v} = \mathbf{w}$?

No, $\mathbf{u} \times 2\mathbf{u} = 0$ and $\mathbf{u} \times 3\mathbf{u} = 0$ but $2\mathbf{u} \neq 3\mathbf{u}$.

[3 marks] 5. For the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ determine

$$\big|(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})\big|$$

First, $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$ which in this case is

$$|\mathbf{a}|^2 = 4 + 9 + 25 = 38 \quad \text{and} \quad |\mathbf{b}|^2 = 9 + 1 + 4 = 14$$

Therefore,

$$\big|(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})\big| = |38 - 14| = 24.$$

[3 marks] 6. Find a vector parallel to the line of intersection of the two planes given by

$$2x - y + z = 1 \quad \text{and} \quad 3x + y + z = 2$$

The two planes have normals $\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. The line of intersection must be perpendicular to both of these normals.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} = -2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

This gives us the direction of the line, we also need to find a point lying on it, that is a point of intersection of the planes. This is the same as find a solution to the linear system

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{array}\right)$$

This has a solution $x = 1$, $y = 0$, $z = -1$. Thus the line of intersection is

$$(1 - 2t)\mathbf{i} + t\mathbf{j} + (-1 + 5t)\mathbf{k}.$$

[3 marks] 7. Consider the point $P = (3, -1, 2)$.

(a) Determine the distance between P and $(1, 4, -1)$.

(b) Determine the normal vector to the plane $x - 4y + 2z = 1$.

(c) Determine the minimal distance between P and the plane $x - 4y + 2z = 1$.

(a) The distance between P and $(1, 4, -1)$ is equal to

$$\sqrt{(1-3)^2 + (4-(-1))^2 + (-1-2)^2} = \sqrt{4 + 25 + 9} = \sqrt{38}$$

(b) The normal vector to the plane $x - 4y + 2z = 1$ is

$$\mathbf{n} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

(c) The minimal distance between P and the plane $x - 4y + 2z = 1$ is given by the distance between P and the point of intersection of the plane and the normal through P. The line of the normal through P is

$$(3 + t)\mathbf{i} + (-1 - 4t)\mathbf{j} + (2 + 2t)\mathbf{k}$$

This line intersects the plane at the point

$$(3 + t) - 4(-1 - 4t) + 2(2 + 2t) = 1$$

which is equivalent to

$$21t = -10 \quad \text{that is} \quad t = -10/21$$

Thus the point of intersection is

$$(3 - 10/21)\mathbf{i} + (-1 + 40/21)\mathbf{j} + (2 - 20/21)\mathbf{k} = 53/21\mathbf{i} + 19/21\mathbf{j} + 22/21\mathbf{k}$$

The distance between this point and P is

$$\sqrt{(53/21 - 3)2 + (19/21 - (-1))^2 + (22/21 - 2)^2} = \frac{10}{\sqrt{21}}.$$

[2 marks] 8. Consider the function $f\colon \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (2t, e^t)$.

(a) Determine $f(0)$, $f(1)$ and $f(-1)$.

(b) Is f linear?

(a) $f(0) = (0,1)$, $f(1) = (2,e)$, and $f(-1) = (-2, e^{-1})$.
(b) No, f is not linear. $f(0+1) = f(1) = (2,e)$. But $f(0)+f(1) = (0,1)+(2,e) = (2,1+e)$. Therefore $f(0+1) \neq f(0) + f(1)$.

[2 marks] 9. Form all possible composite functions from the four functions below,

$$e\colon \mathbb{R} \to \mathbb{R}^2\,, \quad f\colon \mathbb{R}^3 \to \mathbb{R}\,, \quad g\colon \mathbb{R}^2 \to \mathbb{R}^5\,, \quad h\colon \mathbb{R}^4 \to \mathbb{R}^3\,.$$

Also give the domain and codomain for each composite function.

The possible composite functions are:

$$g \circ e\colon \mathbb{R} \to \mathbb{R}^5\,, \qquad f \circ h\colon \mathbb{R}^4 \to \mathbb{R}\,, \qquad e \circ f\colon \mathbb{R}^3 \to \mathbb{R}^2$$
$$g \circ e \circ f\colon \mathbb{R}^3 \to \mathbb{R}^5\,, \qquad e \circ f \circ h\colon \mathbb{R}^4 \to \mathbb{R}^2$$
$$g \circ e \circ f \circ h\colon \mathbb{R}^4 \to \mathbb{R}^5$$

[2 marks] Bonus question: If A is an $n \times n$ matrix such that $A^3 = 0$, show that $I - A$ is invertible.

We have the identity

$$(I - A)(I + A + A^2) = I + A + A^2 - (A + A^2 + A^3) = I - A^3$$

if $A^3 = 0$ then

$$(I - A)(I + A + A^2) = I$$

and so $I - A$ is invertible.
