MATH349 – Assignment 4 – Solutions

[3 marks] 1. Express the following points, which are given in Cartesian coordinates, in polar coordinates:

$$P = (1, 1)\ , \quad Q = (-3, 4)\ , \quad R = (0, -7)\ .$$

For P, we get $r = \sqrt{2}$ and $\theta = \pi/4$. For Q we get $r = \sqrt{9+16} = 5$ and $\theta = \tan^{-1}(-3/4) = 2.214$ noting that θ lies in the second quadrant. For R we get $r = 7$ and $\theta = 3\pi/2$.

[4 marks] 2. Find the arclength of the following curves

(a) $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$ between $t = 0$ and $t = 2\pi$.

(b) $X(t) = (e^{3t}, e^{-3t}, 3\sqrt{2}t)$ between $t = 0$ and $t = \dfrac{1}{3}$.

(a) Differentiating gives

$$\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$$

The arclength between $t = 0$ and $t = 2\pi$ is given by

$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t} dt = \int_0^{2\pi} \sqrt{2 - 2\cos t} dt$$

To evaluate this integral it helps to recall $1 - \cos t = 2\sin^2(t/2)$. With this simplification, the integral becomes

$$\int_0^{2\pi} 2\sin(t/2)dt = \left[4\cos(t/2)\right]_0^{2\pi} = 8$$

Thus the arclength is 8.

(b) Differentiating gives

$$\dot{X}(t) = (3e^{3t}, -3e^{-3t}, 3\sqrt{2})$$

The arclength between $t = 0$ and $t = 1/3$ is

$$\int_0^{1/3} \sqrt{9e^{6t} + 9e^{-6t} + 18} dt = 3\int_0^{1/3} \sqrt{e^{6t} + 2 + e^{-6t}} dt$$

It helps to notice $e^{6t} + 2 + e^{-6t} = (e^{3t} + e^{-3t})^2$, so the integral becomes

$$3\int_0^{1/3} e^{3t} + e^{-3t} dt = 3\left[\frac{1}{3}\left(e^{3t} - e^{-3t}\right)\right]_0^{1/3}$$
$$= e - e^{-1}$$

[3 marks] 3. The Folium is defined as the set of points in $\mathbb{R}^2$ which satisfy the equation

$$x^3 + y^3 = 3axy$$

where a is a fixed constant.

(a) Roughly, draw the Folium in the case that $a = 1$.

(b) Draw a line of slope t joining $(0, 0)$ to another point P on the Folium.

(c) Find rational functions $x = u(t)$ and $y = v(t)$ which parametrize the Folium by finding the second point of intersection of the line $y = tx$ through the origin.

(c) Substituting $y = tx$ into $x^3 + y^3 = 3axy$ gives

$$x^3 + t^3x^3 = 3atx^2 \quad \text{which implies} \quad x^2\big((1+t^3)x - 3at\big) = 0$$

Since we want the second point of intersection of the line on the Folium, we may discard the $x = 0$ solution, so

$$(1+t^3)x - 3at = 0 \quad \text{that is} \quad x = \frac{3at}{1+t^3}$$

Since $y = tx$ we get $y = \dfrac{3at^2}{1+t^3}$.

[3 marks] 4. The Roses of Grandi are defined by the polar equation

$$r = a\cos n\theta$$

for integer values of n and a is a fixed constant.

(a) Show that the "rose" for $n = 1$ is a circle.
(b) Show that the "rose" for $n = 2$ satisfies the equation

$$(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$$

(c) Roughly, plot the set of points which satisfy the equation in (b).

(a) When $n = 1$, we have $r = a\cos\theta$, thus $r^2 = ra\cos\theta = ax$. Since $r^2 = x^2 + y^2$ also, we find

$$x^2 + y^2 = ax \quad \text{which transforms to} \quad (x - a/2)^2 + y^2 = a^2/4$$

which is the equation of a circle with radius a centred at $(a/2, 0)$.
(b) When $n = 2$ we have $r = a\cos 2\theta$, so

$$\begin{aligned} r^3 = ar^2\cos 2\theta &= ar^2(\cos^2\theta - \sin^2\theta) \\ &= a(x^2 - y^2) \end{aligned}$$

Thus, $r^6 = (x^2 + y^2)^3 = a^2(x^2 - y^2)^3$.

[4 marks] 5. Verify that $\mathbf{r}(t) = \mathbf{r}_0\cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)$ satisfies the initial value problem

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}\,, \quad \mathbf{r}'(0) = \mathbf{v}_0\,, \quad \mathbf{r}(0) = \mathbf{r}_0\,.$$

(It is the unique solution.) Describe the path $\mathbf{r}(t)$. What is the path if $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$?

Differentiating gives

$$\frac{d\mathbf{r}}{dt} = -\omega\mathbf{r}_0\sin(\omega t) + \mathbf{v}_0\cos(\omega t)$$

and differentiating again

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= -\omega^2\mathbf{r}_0\cos(\omega t) - \omega\mathbf{v}_0\sin(\omega t)\\ &= -\omega^2(\mathbf{r}_0\cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)\\ &= -\omega^2\mathbf{r}\end{aligned}$$

Also,

$$\mathbf{r}(0) = \mathbf{r}_0 \quad \text{and} \quad \mathbf{v}(0) = \mathbf{v}_0.$$

The main point to notice about the path of $\mathbf{r}(t)$ is that it is closed because both $\sin(\omega t)$ and $\cos(\omega t)$ have a period length of $2\pi/\omega$. However, the shape of the closed curve can vary. For instance, if $\mathbf{v}_0$ is parallel to $\mathbf{r}_0$ (or if either of them is zero) then $\mathbf{r}$ is a line segment in the direction $\mathbf{r}_0$. If $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$ then we obtain either a circle or an ellipse. We get a circle only in the case $|\mathbf{r}_0| = |\mathbf{v}_0/\omega|$.

[3 marks] 6. Let $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$ and let $\theta(t)$ be the angle which the tangent line at a given point of the curve makes with the z-axis. Show that $\cos\theta(t)$ is the constant $\dfrac{b}{\sqrt{a^2+b^2}}$.

The direction of the tangent line is the direction of the velocity vector,

$$\mathbf{v}(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}.$$

The direction the z-axis is $\mathbf{k}$. The angle θ between these two lines is given by

$$\begin{aligned}\cos\theta &= \frac{\mathbf{k}\bullet\mathbf{v}(t)}{|\mathbf{k}||\mathbf{v}(t)|}\\ &= \frac{b}{\sqrt{a^2+b^2}}\end{aligned}$$

[2 marks] 7. Show that if $\kappa(s) = c$ is a positive constant and $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circle. (Note that here the curve is parametrized in terms of arc length.)

The long answer is to show that $\tau(s) = 0$ implies $\mathbf{r}$ lies in a plane and then from positive constant curvature deduce that the curve is a circle. The short answer is to note that (as in class) the parametrization of a circle is

$$\mathbf{u}(s) = a\sin(s/a)\mathbf{i} + a\cos(s/a)\mathbf{j} + 0\mathbf{k}$$

which has positive constant curvature and zero torsion. The theorem on space curves says that if $\mathbf{r}$ has the same curvature and zero torsion then it is just a translation or a rotation of this circle. Since the translation and rotation of a circle is still a circle, we must have that $\mathbf{r}$ is a circle.

[3 marks] 8. Find the radius of curvature of the curve

$$\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

at the point $t = 1$.

The velocity vector is

$$\mathbf{v}(t) = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

and the acceleration vector is

$$\mathbf{a}(t) = 6t\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}.$$

At $t = 1$, we find $\mathbf{v}(1) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{a}(1) = 6\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$ and $v(1) = \sqrt{9+4+1} = \sqrt{14}$. The curvature is given by

$$\kappa(1) = \frac{\left|\mathbf{v}(1) \times \mathbf{a}(1)\right|}{v(1)^3}$$

and

$$\mathbf{v}(1) \times \mathbf{a}(1) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 6 & 2 & 0 \end{pmatrix} = -2\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$$

so the curvature is

$$\kappa(1) = \frac{\sqrt{4+36+36}}{14^{3/2}} = \frac{\sqrt{76}}{14^{3/2}}$$

Therefore the radius of curvature is $\dfrac{14^{3/2}}{\sqrt{76}} = \dfrac{7\sqrt{14}}{\sqrt{19}}$.

[2 marks] Bonus question: (This question is about Kepler's Laws which can be found in section 11.6 of Adams.) Show that the orbital speed of a planet is constant if and only if the orbit is circular. (Hint: Use the conservation of energy identity.)

The conservation of energy equation gives us

$$\dot{r}^2 r^2 + h^2 - 2rk - 2Er^2 = 0$$

which is equal to

$$(\dot{r}^2 - 2E)r^2 - 2kr + h^2 = 0$$

The orbit is then circular if and only if the coefficients of the equation in r are constants. This is equivalent to $\dot{r}$ begin constant, that is the planet must have constant orbital speed.