MATH349 – Assignment 5 – Solutions

[3 marks] 1. Determine the Frenet frames (i.e. $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$) for the curve

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

at the point $(1, 1, 1)$.

The velocity vector is

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k} \quad \text{which gives} \quad \mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

and the acceleration vector is

$$\mathbf{a}(t) = 2\mathbf{j} \quad \text{which gives} \quad \mathbf{a}(1) = 2\mathbf{j}.$$

The tangent vector at $t = 1$ is then

$$\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

The unit binormal vector at $t = 1$ is

$$\hat{\mathbf{B}} = \frac{\mathbf{v}(1) \times \mathbf{a}(1)}{|\mathbf{v}(1) \times \mathbf{a}(1)|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$$

We can then find the normal vector at $t = 1$ as

$$\begin{aligned}\hat{\mathbf{N}} &= \hat{\mathbf{B}} \times \hat{\mathbf{T}} \\ &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \\ &= \frac{1}{\sqrt{3}}(-\mathbf{i} + \mathbf{j} - \mathbf{k})\end{aligned}$$

[3 marks] 2. Determine the tangent plane, normal vector and normal line for the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = e^{2xy} + 3x^3y^2$$

at the point $(1, 2)$.

The normal vector is

$$\begin{aligned}\mathbf{n} &= \left.\frac{\partial f}{\partial x}\right|_{(1,2)} \mathbf{i} + \left.\frac{\partial f}{\partial y}\right|_{(1,2)} \mathbf{j} - \mathbf{k} \\ &= \left[2ye^{2xy} + 9x^2y^2\right]_{(1,2)} \mathbf{i} + \left[2xe^{2xy} + 6x^3y\right]_{(1,2)} \mathbf{j} - \mathbf{k} \\ &= (4e^4 + 36)\mathbf{i} + (2e^4 + 12)\mathbf{j} - \mathbf{k}\end{aligned}$$

Therefore the normal line is

$$\begin{aligned} x &= 1 + (4e^4 + 36)t \\ y &= 2 + (2e^4 + 12)t \\ z &= e^4 + 12 - t \end{aligned}$$

and the tangent plane is

$$(4e^4 + 36)(x - 1) + (2e^4 + 12)(y - 2) - (z - (e^4 + 12)) = 0.$$

[2 marks] 3. Draw the graphs of the following functions

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto 4 - x^2$.

(b) $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^2 + y^2 - 2y$.

[2 marks] 4. Draw the level curves of the following functions

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto (x + 1)(y + 3)$.

(b) $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x^2/y$.

[2 marks] 5. Find all first partial derivatives of $h\colon \mathbb{R}^3 \to \mathbb{R}$ where $h(x, y, z) = x^2y^4z^7$. Evaluate each of them at the point $(1, 0, -1)$.

The first partial derivatives of h are

$$\frac{\partial h}{\partial x} = 2xy^4z^7 , \quad \frac{\partial h}{\partial y} = 4x^2y^3z^7 , \quad \frac{\partial h}{\partial z} = 7x^2y^4z^6$$

At the point $(1, 0, -1)$ they are

$$\left.\frac{\partial h}{\partial x}\right|_{(1,0,-1)} = -2 , \quad \left.\frac{\partial h}{\partial y}\right|_{(1,0,-1)} = -4 , \quad \left.\frac{\partial h}{\partial z}\right|_{(1,0,-1)} = 7 .$$

[2 marks] 6. Compute the matrix of first partial derivatives for the function

$$\begin{aligned} &f\colon \mathbb{R}^2 \to \mathbb{R}^4 \\ &(x, y) \mapsto (x + y, x - y, 2x + y^2, y) \end{aligned}$$

The derivative matrix of f is

$$Df(x, y) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2y \\ 0 & 1 \end{pmatrix}$$

[3 marks] 7. For the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

determine $f_x(0,0)$, $f_y(0,0)$, $f_{xx}(0,0)$, $f_{xy}(0,0)$, $f_{yx}(0,0)$ and $f_{yy}(0,0)$.

First,

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \end{aligned}$$

Next, when $(x,y) \neq (0,0)$ we have

$$\begin{aligned} f_x(x,y) &= \frac{(x^2+y^2)(3x^2y-y^3)-2x^2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \\ f_y(x,y) &= \frac{(x^2+y^2)(x^3-3xy^2)-2xy^2(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} \end{aligned}$$

Hence,

$$\begin{aligned} f_{xx}(0,0) &= \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \\ f_{xy}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h^5/h^4-0}{h} = -1 \\ f_{yx}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{h^5/h^4-0}{h} = 1 \\ f_{yy}(0,0) &= \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \end{aligned}$$

Notice that this is an example of $f_{xy} \neq f_{yx}$.

[3 marks] 8. Consider the function $F\colon \mathbb{R}^3 \to \mathbb{R}$ given by $F(x,y,z) = z\sin(y/x)$ and the transformation

$$x = 3r^2 + 2s\ , \quad y = 4r - 2s^3\ , \quad z = 2r^2 - 3s^2\ .$$

Determine $\dfrac{\partial F}{\partial r}$ and $\dfrac{\partial F}{\partial s}$.

We have

$$DF(x,y,z) = \begin{pmatrix} \dfrac{-zy}{x^2}\cos(y/x) & \dfrac{z}{x}\cos(y/x) & \sin(y/x) \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} \end{pmatrix} = \begin{pmatrix} 6r & 2 \\ 4 & -6s^2 \\ 4r & -6s \end{pmatrix}$$

By the chain rule,

$$\begin{aligned} \begin{pmatrix} \frac{\partial F}{\partial r} & \frac{\partial F}{\partial s} \end{pmatrix} &= \begin{pmatrix} \frac{-zy}{x^2}\cos(y/x) & \frac{z}{x}\cos(y/x) & \sin(y/x) \end{pmatrix} \begin{pmatrix} 6r & 2 \\ 4 & -6s^2 \\ 4r & -6s \end{pmatrix} \\ &= \Bigg(\cos(y/x)\left(\frac{-6rzy}{x^2} + 4\frac{z}{x}\right) + 4r\sin(y/x) \\ &\qquad\qquad \cos(y/x)\left(\frac{-2zy}{x^2} + \frac{-6s^2 z}{x}\right) - 6s\sin(y/x) \Bigg) \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial F}{\partial r} = \cos\left(\frac{4r-2s^3}{3r^2+2s}\right) \frac{2r^2-3s^2}{(3r^2+2s)^2}\Big\{ -6r(2r^2-3s^2)(4r-2s^3) + 4(3r^2+2s)\Big\} \\ + 4r\sin\left(\frac{4r-2s^3}{3r^2+2s}\right) \end{aligned}$$

and

$$\frac{\partial F}{\partial s} = \cos\left(\frac{4r-2s^3}{3r^2+2s}\right) \frac{2r^2-3s^2}{(3r^2+2s)^2}\Big\{ -2(4r-2s^3) - 6s^2\Big\} - 6s\sin\left(\frac{4r-2s^3}{3r^2+2s}\right)$$

[2 marks] 9. Find the derivative matrix and the Jacobian of the transformation $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ where

$$(r, \theta, \phi) \mapsto (r\sin\theta\sin\phi, r\cos\theta\sin\phi, r\cos\phi)$$

The co-ordinates (r, θ, ϕ) are known as the spherical co-ordinates for $\mathbb{R}^3$.

The derivative matrix for S is

$$DS(r,\theta,\phi) = \begin{pmatrix} \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

Therefore the Jacobian of the transformation is

$$\begin{aligned} \frac{\partial(S_1,S_2,S_3)}{\partial(r,\theta,\phi)} &= \cos\phi \det\begin{pmatrix} r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi & r\cos\theta\cos\phi \end{pmatrix} - r\sin\phi\begin{pmatrix} \sin\theta\sin\phi & r\cos\theta\sin\phi \\ \cos\theta\sin\phi & -r\sin\theta\sin\phi \end{pmatrix} \\ &= \cos\phi(r^2\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta) - r\sin\phi(-r\sin^2\phi)(-\cos^2\theta - \sin^2\theta) \\ &= r^2\cos^2\phi\sin\phi + r^2\sin^3\phi \\ &= r^2\sin\phi \end{aligned}$$

[3 marks] 10. Show that the function

$$u\colon \mathbb{R}^3 \to \mathbb{R}$$
$$(x, y, z) \mapsto \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

satisfies Laplace's partial differential equation, namely

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The first partial derivatives are

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\ , \quad \frac{\partial u}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\ , \quad \frac{\partial u}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}$$

The second partial derivatives are

$$\frac{\partial^2 u}{\partial x^2} = \frac{-(x^2 + y^2 + z^2)^{3/2} + 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{-(x^2 + y^2 + z^2)^{3/2} + 3y^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}$$
$$\frac{\partial^2 u}{\partial z^2} = \frac{-(x^2 + y^2 + z^2)^{3/2} + 3z^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}$$

Thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3(x^2 + y^2 + z^2)^{3/2} + 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} = 0$$

that is, u satisfies Laplace's partial differential equation.

[2 marks] Bonus question: Use the ε–δ definition of a limit to show that

$$\lim_{(x,y)\to(2,1)} (3x + y^2) = 7$$

We want to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\big|(x, y) - (2, 1)\big| < \delta \quad \text{implies} \quad \big|3x + y^2 - 7\big| < \varepsilon.$$

First, $\big|(x, y) - (2, 1)\big| < |x - 2| + |y - 1|$. Thus, $|x - 2| < \delta/2$ and $|y - 1| < \delta/2$ implies $\big|(x, y) - (2, 1)\big| < \delta$. Furthermore, if $|y - 1| < \delta/2$ then $-\delta/2 + 1 < y < 1 + \delta/2$. Hence

$$\delta^2/4 - \delta/2 + 1 < y^2 < 1 + \delta/2 + \delta^2/4 \quad \text{and} \quad -3\delta/2 + 6 < 3x < 3\delta/2 + 6$$

Adding gives

$$\delta^2/4 - 2\delta + 7 < 3x + y^2 < 7 + 2\delta + \delta^2/4.$$

Since $\delta^2 > 0$ we get

$$-2\delta < 3x + y^2 - 7 < 2\delta + \delta^2/4.$$

Next, if $\delta \leqslant 1$ we may take

$$-3\delta < 3x + y^2 - 7 < 3\delta \quad \text{which is} \quad |3x + y^2 - 7| < 3\delta$$

Thus we need only take $\delta = \varepsilon/3$ or $\delta = 1$ when $\varepsilon \geqslant 3$ to obtain

$$|3x + y^2 - 7| < \varepsilon.$$

Hence,

$$\lim_{(x,y)\to(2,1)} (3x + y^2) = 7.$$