[3 marks ] 1. Give the first 5 terms of the sequence

$$u_{n+2} = 3u_{n+1} - u_n$$

where $u_0 = 1$, $u_1 = 2$.

The first five terms are: $u_0 = 1$, $u_1 = 2$, $u_2 = 5$, $u_3 = 13$, $u_4 = 34$.

[4 marks ] 2. (a) State the Integral Test

(b) Show that $\sum_{n=1}^{\infty} 1/n^2$ is convergent.

(c) State a version of the Comparison Test.

(d) Show that

$$1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots$$

is convergent.

(a) Suppose that $a_n = f(n)$ where $f$ is positive, nonincreasing and continuous on $[N, \infty)$ for some $N > 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_N^{\infty} f(t)dt$ converges.

(b) Let $f(t) = 1/t^2$. Then $f$ is positive and nonincreasing (since $(t+1)^2 > t^2$ implies $1/(t+1)^2 < 1/t^2$). It is also continuous on $[1, \infty)$. Since

$$\int_1^{\infty} \frac{1}{t^2} = \left[\frac{-1}{t}\right]_1^{\infty} = 1$$

converges we have that $\sum_{n=1}^{\infty} 1/n^2$ is also convergent.

(c) Let $\{a_n\}$ and $\{b_n\}$ be sequences which satisfy $0 \leq a_n \leq b_n$. Then,

- if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
- if $\sum_{n=1}^{\infty} a_n$ diverges to infinity, then so does $\sum_{n=1}^{\infty} b_n$.

(d) The series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots$$

has $n$-th term $1/n(n+1)$. Thus the $n$-th term is less than $1/n^2$. Since $\sum 1/n^2$ is convergent we have that the above series is also convergent by the Comparison Test. Thus

$$1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots$$

is also convergent.

[3 marks ] 3. Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to be convergent.

The series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums $S_k = \sum_{n=1}^{k} a_n$ is convergent.

[4 marks ] 4. Show that the sequence $s_n = \frac{2n-7}{3n-2}$ is bounded.

Note that $2n-7>0$ for $n>3$ and $3n-2>0$ for $n>0$, hence $s_n>0$ for $n>3$, that is $s_n$ is bounded below by 0. Next $2n-7<3n-2$ for all $n>0$ hence $s_n<1$ for all $n>0$, that is $s_n$ is bounded above by 1. Since $s_n$ is bounded above and below it is bounded.

[4 marks ] 5. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ is conditionally convergent, absolutely convergent, or divergent.

First, $\log n < n$ implies $1/\log n > 1/n$ and $\sum 1/n$ is divergent, so $\sum 1/\log n$ is divergent by the Comparison Test. Hence the series is not absolutely convergent. The sequence $1/\log n$ is positive (since $\log n > 0$ for $n > 1$, decreasing and convergent to 0 (since $\log n \to \infty$.) Hence by the alternating series test, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ is conditionally convergent.

[3 marks ] 6. Give the Taylor series about 2 for $e^x$.

Let $t = x - 2$. Then

$$\begin{aligned} e^t &= 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \\ e^{x-2} &= 1 + \frac{(x-2)}{1!} + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-2)^4}{4!} + \cdots \\ e^x &= e^2 + \frac{e^2(x-2)}{1!} + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!} + \frac{e^2(x-2)^4}{4!} + \cdots \end{aligned}$$

and this expansion is valid for all $x$.

[4 marks ] 7. Find the value of the series given by

$$\frac{1}{2\cdot 2!}+\frac{1}{4\cdot 3!}+\frac{1}{8\cdot 4!}+\frac{1}{16\cdot 5!}+\frac{1}{32\cdot 6!}+\dots$$

From the Taylor series for $e^x$ we find

$$\begin{aligned} e^{1/2} &= 1+\frac{1}{2}+\frac{1}{2^2 2!}+\frac{1}{2^3 3!}+\frac{1}{2^4 4!}+\frac{1}{2^5 5!}+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{4\cdot 2!}+\frac{1}{8\cdot 3!}+\frac{1}{16\cdot 4!}+\frac{1}{32\cdot 5!}+\cdots \end{aligned}$$

Hence we can write

$$2e^{1/2} = 2+1+\frac{1}{2\cdot 2!}+\frac{1}{4\cdot 3!}+\frac{1}{8\cdot 4!}+\frac{1}{16\cdot 5!}+\cdots$$

so

$$2e^{1/2}-3 = \frac{1}{2\cdot 2!}+\frac{1}{4\cdot 3!}+\frac{1}{8\cdot 4!}+\frac{1}{16\cdot 5!}+\cdots$$

Thus, the value of the series is $2e^{1/2}-3$.

[4 marks ] 8. Find the equation of the plane passing through the points

$$(0,1,-1)\ ,\quad (3,-1,2)\ ,\quad (1,2,4)\ .$$

The vector joining $(0,1,-1)$ and $(3,-1,2)$ has direction $(3,-2,3)$. The vector joining $(0,1,-1)$ and $(1,2,4)$ has direction $(1,1,5)$. The normal to the plane is the cross product of these two vectors,

$$\mathbf{n} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 3 \\ 1 & 1 & 5 \end{pmatrix} = -13\mathbf{i}-12\mathbf{j}+5\mathbf{k}$$

Thus the equation of the plane is

$$-13x-12(y-1)+5(z+1)=0.$$

[4 marks ] 9. Find the unit vector perpendicular to

$$\mathbf{i}-\mathbf{j}+2\mathbf{k} \quad \text{and} \quad 2\mathbf{i}-3\mathbf{k}$$

The vector perpendicular to the given ones is given by the cross product,

$$\mathbf{v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix} = 3\mathbf{i}+7\mathbf{j}+2\mathbf{k}$$

This vector has length $\sqrt{9+49+4} = \sqrt{62}$. Therefore the unit vector parallel to the two in question is

$$\frac{1}{\sqrt{62}}(3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$$

[4 marks ] 10. Find the velocity vector and speed at time $t$ of

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

The velocity vector is

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}.$$

The speed is

$$|\mathbf{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}.$$

[3 marks ] 11. If $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$ show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$.

If $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$ then $\mathbf{w} = -(\mathbf{u} + \mathbf{v})$, so

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \mathbf{v} \times (-\mathbf{u} - \mathbf{v}) \\ &= (\mathbf{v} \times -\mathbf{u}) + (\mathbf{v} \times -\mathbf{v}) \\ &= \mathbf{u} \times \mathbf{v} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{w} \times \mathbf{u} &= (-\mathbf{u} - \mathbf{v}) \times \mathbf{u} \\ &= (-\mathbf{u} \times \mathbf{u}) + (-\mathbf{v} \times \mathbf{u}) \\ &= \mathbf{u} \times \mathbf{v} \end{aligned}$$

So $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$.