University of Calgary
Faculty of Science
Department of Mathematics and Statistics

Math 349 Spring 2006

Worksheet 8

1. Use the chain rule to find $\frac{dz}{dt}$ at the specified value of t.

a. $z = x^2 + y^2, \quad x = t^2 + 1, \quad y = 2 - t^2; \quad at \quad t = 2.$

b. $z = x \cos(xy), \quad x = 2t^2, \quad y = \frac{1}{t}; \quad at \quad t = \frac{\pi}{4}.$

2. Use the chain rule to find $\frac{dz}{dt}$. Express your answers in terms of t.

a. $z = \ln\left(\frac{y}{x}\right), \quad x = \sin t, \quad y = \cos t.$

b. $z = e^{-x^2/y}, \quad x = \sinh t, \quad y = 1 + \cosh t.$

3. Let $z = f(x, y), \quad x = t^2, \quad y = \ln t$. Find $\frac{d^2z}{dt^2}$. You may assume all suitable differentiability conditions are met.

4. Let $z = f(x, y), \quad x = r\cos\theta, \quad y = r\sin\theta$. Prove that

$$\frac{\partial^2 z}{\partial r^2} = f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta;$$

$$r\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial\theta^2} = r^2\sin^2\theta\, f_{xx} - 2r^2\sin\theta\cos\theta\, f_{xy} + r^2\cos^2\theta\, f_{yy}.$$

5. If $w = xy^2z^3, \quad x = e^t, \quad y = e^t, \quad and \quad z = t\,e^t$, find $\frac{dz}{dt}$.

6. Find $\frac{\partial z}{\partial u}, \quad \frac{\partial z}{\partial v}$ at the point specified.

a. $z = x^2 - 2xy, \quad x = \frac{u}{v}, \quad y = uv; \quad at \quad (u, v) = (2, -1).$

b. $z = \sin(x + y), \quad x = u^2 - v^2, \quad y = 2uv; \quad at \quad (u, v) = \left(\sqrt{\pi}, \frac{\sqrt{\pi}}{2}\right)$

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Math 349 Spring 2006

Worksheet 8

7. Let $z = x^2 + 2xy - y^2$, $x = \cosh u + \sinh v$, $y = \cosh u - \sinh v$. Find $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$.

8. If $w = \sqrt{xyz}$, $x = t^2 s$, $y = \dfrac{t}{s}$, $z = \dfrac{s^2}{t}$, $ts > 0$. Find $\dfrac{\partial w}{\partial t}$, $\dfrac{\partial w}{\partial s}$.

9. Let $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. Verify that $z_x^{\,2} + z_y^{\,2} = z_r^{\,2} + \dfrac{1}{r^2} z_\theta^{\,2}$.

10. Let $z = f\left(\dfrac{x}{y}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$. Hint: let $t = \dfrac{x}{y}$.

11. Let $z = f(2x - 3y, 3y - 2x)$, show that $3\dfrac{\partial z}{\partial x} + 2\dfrac{\partial z}{\partial y} = 0$.

12. Assume the given equation defines z as a differentiable function of x and y on some domain. Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

a. $x^2 yz - 2y^2 - 3z^2 = 0$

b. $z - xy \ln z = 1$

c. $\sin(xz) + \ln\cos(yz) = 1$

d. $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{y}{z}\right) = 2$

13. Find the directional derivative of f at the given point in the direction of the given vector:

a. $f(x,y) = 2x^2 y - 3xy^2$ *at* $(3,1)$; $\mathbf{u} = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$

b. $f(x,y) = x\,e^y - y\,e^x$ *at* $(0,0)$; $\mathbf{v} = \langle 8, -1 \rangle$

c. $f(x,y,z) = x^2 - 2y^2 + 3z^2$ *at* $(2,0,-1)$; $\mathbf{u} = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3} \right\rangle$

University of Calgary
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Math 349 Spring 2006

Worksheet 8

d. $f(x,y,z) = x \cosh(y + z) \quad at \quad (3,2,-1); \quad \boldsymbol{v} = \langle 1, 1, -1 \rangle$

14. Find the directional derivative of $f(x, y) = \ln\left(\sqrt{x^2 - 2y^2}\right)$ *at* $(2, 1)$ in the direction of the vector from (2, 1) to (5, -3).

15. In each case find the unit vector **u** for which $D_u f$ is a maximum, and give this maximum value.

a. $f(x,y,z) = \ln\left(\dfrac{x + 2y}{z^3}\right); \quad at \quad (5, -2, 3).$

b. $f(x,y) = \sqrt{\dfrac{x - y}{x + y}} \quad at \quad (5,4).$

16. Repeat exercise 15 for $D_u f$ a minimum.

17. In what direction from the point (1, -1) is the instantaneous rate of change of $f(x,y) = 2x^2 + 2xy - 3y^2$ equal to 2 ? In what direction from the point (1, -1) does this function increase most rapidly? What is this most rapid rate of change?

18. Find the equation of the tangent plane and normal line to the given surface at the specified point:

a. $xy + 2yz + 3xz = 16; \quad at \quad (4,-2,3).$

b. $z^2 = 3x^2 + 4y^2; \quad at \quad (-2,1,4).$

c. $y = \ln\left(\dfrac{x + 2y}{y + 2z}\right) - 1 ; \quad at \quad (3,-1,1).$

d. $\sin\left(\dfrac{x}{y}\right) + \cos\left(\dfrac{y}{z}\right) = 0 ; \quad at \quad \left(\pi, 1, \dfrac{2}{\pi}\right).$

19. Find the point on the surface $z = 3x^2 + 4y^2$ on which the tangent plane is perpendicular to the line through the points A(1,-2,4) and B(-2,0,3).