MATH 353 Handout #1 Solutions

A

1. Find the boundary ∂S .Is the set S closed,open?

   (a) $S = \left\{(x, y); \frac{|x|}{|y|} \leq 1\right\}$.

   (b) $S = \{(x, y); y - 2x = 1, 1 \leq y \leq 3\}$

   (c) $S = \{$ all irrational numbers between 0 and 1$\} \subset R$

2. Classify all critical points of the function $f(x, y) = xye^{-2x^2 - \frac{y^4}{4}}$.

Solutions

For 1a)

the set is symmetrical in x and y,and $y \neq 0$,
let's investigate for $x \geq 0, y > 0....1st$ quadrant $x \leq y$
points are on and above the line $y = x$
now reflection in both axes,the origin is excluded
we can see that all points on both lines $y = x$ and $y = -x$
are the boundary points included the origin
Thus $\partial S = \{(x, y); y = x$ and $y = -x\}$
all points on the boundary except (0,0) are in the set,but not all
so the set is NOT closed, some are in so the set cannot be open.

For 1b)

the set is a line segment incl.the ends $A(0, 1)$ and $B(1, 3)$
any point on the line is a boundary point
since"close by" are points from the segment and also from the complement $R^2 - S$
$\partial\ S = S$ and the set is closed

For 1c)

the boundary is the closed interval $[0, 1]$ since in any small interval
we can find both rational and irrational number
$S \subset \partial S$ but $S \neq \partial S$ so the set is neither closed nor open.

For 2)

the function is differentiable everywhere,for critical points solve
$f_x = ye^{-2x^2 - \frac{y^4}{4}}(1 - 4x^2) = 0.......y = 0$ or $x = \pm\frac{1}{2}$
$f_y = xe^{-2x^2 - \frac{y^4}{4}}(1 - y^4) = 0......x = 0$ or $y = \pm 1$
so all possible combinations are :
$(0, 0), \left(\pm\frac{1}{2}, 1\right), \left(\pm\frac{1}{2}, -1\right)$ ......5 critical points
$(0, 0)$ is a saddle point since the values of f are positive if $xy > 0$,and negative if $xy < 0$,
$f(0, 0) = 0$ is neither max nor min
$f(\frac{1}{2}, 1) = f(-\frac{1}{2}, -1) = \frac{1}{2}e^{-\frac{3}{4}}$ and $f(\frac{1}{2}, -1) = f(-\frac{1}{2}, 1) = -\frac{1}{2}e^{-\frac{3}{4}}$
so first two could be maxima ,the latter two minima points
to justify use Second Derivative Test
$f_{xx} = ye^{-2x^2 - \frac{y^4}{4}}(-12x + 16x^3) = 4xye^{-2x^2 - \frac{y^4}{4}}(-3 + 4x^2)$ ......A

$f_{xy} = e^{-2x^2 - \frac{y^4}{4}} (1 - 4x^2)(1 - y^4)$ ....B
$f_{yy} = xe^{-2x^2 - \frac{y^4}{4}} (-5y^3 + y^7) = xy^3 e^{-2x^2 - \frac{y^4}{4}} (-5 + y^4)$ C
the discriminant $D = B^2 - AC$
at $(0, 0)$.....$D = 1 > 0$...........**saddle**
at $(\frac{1}{2}, 1)$ and $\left(-\frac{1}{2}, -1\right)$ ....$B = 0, A < 0, C < 0$ so $D < 0$.......l**ocal maxima**
at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$ ..$B = 0, A > 0, C > 0$ so $D < 0$......**.local minima**

B

1. Find the boundary ∂S .Is the set S closed,open?

   (a) $S = \{(x, y); \ln(xy) \le 0\}$.

   (b) $S = \{(x, y); 0 < x^2 + y^2 < 4\}$

   (c) $S = \left\{ \dfrac{n}{3n+1} \right\}_{n=1}^{\infty} \subset R$

2. Classify all critical points of the function $f(x, y) = 2xy^2 - x^2y + 4xy$.

Solutions

For 1a)

ln is defined only for positive numbers so $xy > 0$
to solve $\ln(xy) \le 0$ apply exp. function to both sides
so $xy \le 1$, together $0 < xy \le 1$
the set is in first quadrant under hyperbola $xy = 1$
and in the third quadrant above the hyperbola
therfore **unbounded** the boundary is
$\partial S = \{x = 0 \text{ or } y = 0 \text{ or } xy = 1\}$ ....both axes and hyperbola
part is in (hyperbola),part is out (axes) so the set S is **neither open nor closed.**

For 1b)

we can see that the set is a circular disk without the center and without the circle
Thus $\partial S = \{(x, y); x^2 + y^2 = 4 \text{ and } (0, 0)\}$,
the whole boundary is outside the set ,so S is **open.**

For 1c)

the sequence is convergent to $\frac{1}{3}$ so $\partial S = S \cup \left\{\frac{1}{3}\right\}$
and $\dfrac{1}{3}$ is not in S so S is **neither open nor closed.**

For 2)

the function is differentiable everywhere,for critical points solve
$f_x = 2y^2 - 2xy + 4y = 2y(y - x + 2) = 0$ $\quad y = 0$ or $x - y = 2$
$f_y = 4xy - x^2 + 4x = x(4y - x + 4) = 0$ $\quad x = 0$ or $x - 4y = 4$
So we got 4 critical points
$(0, 0)$ $(0, -2)$, $(4, 0)$and solution of the system $(\frac{4}{3}, -\frac{2}{3})$
For Second Derivative Test
$f_{xx} = -2y....A$ $\quad f_{xy} = 4y - 2x + 4...B$ $\quad f_{yy} = 4x...C$

points	A	B	C	D	conclusion
$(0, 0)$	0	4	0	16	saddle
$(0, -2)$	4	-4	0	16	saddle
$(4, 0)$	0	-4	16	16	saddle
$(\frac{4}{3}, -\frac{2}{3})$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{16}{3}$	$-\frac{48}{9}$	local min

Notice that f has only one local minimum at $(\frac{4}{3}, -\frac{2}{3})$ since $D < 0$ and $A > 0$
and NO maxima.

C

1. Find the boundary ∂S .Is the set S closed,open,bounded?

   (a) $S=\left\{(x,y);\dfrac{x^2}{y}\geq 1\right\}$.Sketch the set in the xy-plane.

   (b) $S=\{(x,y,z);x^2+y^2+2z^2=4\}$.

2. (a) Find all local extrema of the function $f(x,y)=xy\,(4-x-4y)$;

   (b) Find the absolute max/min values of f on the triangle $\triangle\, ABC$ with vertices $A\,(0,0)\,,B\,(0,1)$ and $C\,(1,0)$.

Solutions

For 1a)

the fraction is defined only for $y\neq 0$,
also we can see that y must be positive and $x^2\geq y$
together $x^2\geq y>0$ the set contains all points above the x-axis and below and on the parabola $y=x^2$
the boundary $\partial S=\{\ y=0$ or $y=x^2\}$ ....x- axis and parabola
part is in (parabola) ,part is out (axis) so the set S is **neither open nor closed**.
Also the set is **unbounded.**

For 1b)

we can see that the set is an elliptical shell or ellipsoid
$\partial S=S$, since any point on the shell has "close by" some points on the shell and some points outside and inside the shell
the whole boundary is part of the set ,so S is **closed**.The set is **bounded.** (inside big circle with radius 4).

For 2a) f is differentiable everywhere,for critical points solve
$f_x=y\,(4-x-4y-x)=2y\,(2-2y-x)=0$.......$y=0$ or $x+2y=2$
$f_y=x\,(4-x-4y-4y)=x\,(4-x-8y)=0$......$x=0$ or $x+8y=4$
so all possible combinations are :
$(0,0),(0,1)\,,(4,0)$and $\left(\frac{4}{3},\frac{1}{3}\right)$ ..(by solving the system) 4 critical points
to classify the critical points use the second derivative test:
$f_{xx}=-2y$.....A
$f_{xy}=2\,(2-2y-x-2y)=2\,(2-x-4y)$ ....B
$f_{yy}=-8x$....C the discriminant $D=B^2-AC$
at $(0,0),(0,1)\,,(4,0)$.....$D=B^2>0$...........3 **saddle points**
at $\left(\frac{4}{3},\frac{1}{3}\right)$ $A=\dfrac{-2}{3}$, $B=2\left(2-\frac{4}{3}-\frac{4}{3}\right)=\dfrac{-4}{3}$,$C=\dfrac{-32}{3}$
$D=B^2-AC=\frac{16}{9}\,(1-4)=-\frac{16}{3}<0$ local extremum
and since $A<0$ we'v got **local maximum** at $\left(\frac{4}{3},\frac{1}{3}\right)$.

For 2b)

no critical point from part a) is inside the triangle
so we have to investigate the boundary
$B_1=\{y=0,0\leq x\leq 1\}$ but $f=0$;
$B_2=\{x=0,0\leq y\leq 1\}$ but again $f=0$;
$B_3=\{y=1-x,0\leq x\leq 1\}$

and f on B_3 is equal to $x(1-x)(4-x-4+4x) = 3(x^2-x^3) = g(x)$
for critical points on B_3 solve $g'(x) = 0$
$g'(x) = 3(2x-3x^2) = 3x(2-3x) = 0$ and $x = 0$ or $x = \frac{2}{3}$
therefore besides corners we have a point $\left(\frac{2}{3}, \frac{1}{3}\right)$
and $f\left(\frac{2}{3}, \frac{1}{3}\right) = g(\frac{2}{3}) = \frac{4}{9}$ **is the maximum value ;   0 is the.minimum value.**

D

1. Sketch the set S . Find the boundary ∂S .Is the set S closed,open?

   (a) $S = \left\{(x,y); y \le \dfrac{1}{x}\right\}$

   (b) $S = \left\{(x,y); 9 < \dfrac{1}{x^2+y^2}\right\}$

   (c) $S = \{\sqrt[n]{n}\}_{n=1}^{\infty} \subset R$

2. Find all local extrema i.e.

   Classify all critical points of the function $f(x,y) = 3y^3 - x^2y + x^2$.

Solutions

For 1a)
the function $\frac{1}{x}$ is defined only for $x \neq 0$ so **y-axis is excluded**
for $x > 0$ y could be positive or negative under or on the hyperbola,
for $x < 0$ y must be negative and under or on the hyperbola
the set is **unbounded**
the boundary $\partial S = \left\{x = 0 \text{ or } y = \dfrac{1}{x}\right\}$consists of y-axis and hyperbola
part of the boundary is in (hyperbola) ,part is out (y-axis)
therefore the set S is **neither open nor closed.**
For 1b)
we can see that $x^2 + y^2 \neq 0$ so
the set is a circular disk without the center and without the circle:
$0 < x^2 + y^2 < \frac{1}{3^2}$
Thus $\partial S = \{(x,y); x^2 + y^2 = \frac{1}{9} \text{ or } (0,0)\}$,
the whole boundary is outside the set ,so S is **open and bounded**
For 1c)
the sequence is convergent to 1 and
$\{\sqrt[n]{n}\}_{n=1}^{\infty} = \{1, \sqrt{2}, \sqrt[3]{3}, \sqrt{2}, ...\}$
$\sqrt[1]{1} = 1, \sqrt{2} = 1.41 = \sqrt[4]{4}, \sqrt[3]{3} = 1.44, \sqrt[5]{5} = 1.379, ... \sqrt[10]{10} = 1.26, \sqrt[100]{100} = 1.047...$
so $\partial S = S$ and the set is **closed and bounded**
since

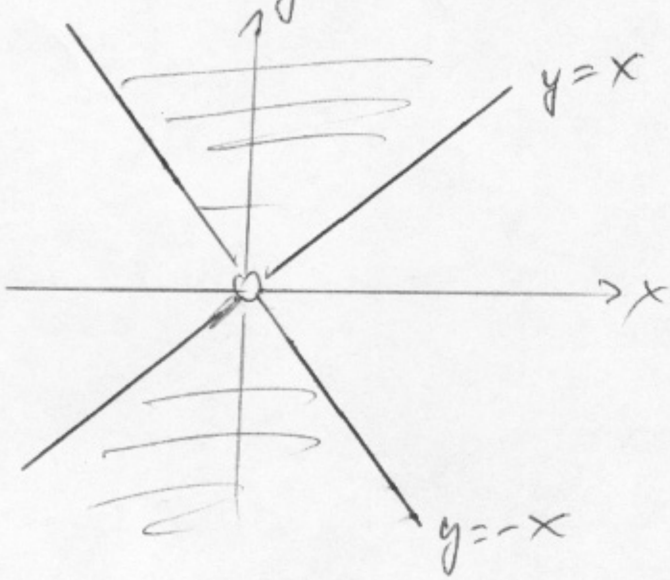
$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1$$

(you can use L'Hop.Rule for the limit of the exponent.)
For 2)
the function is differentiable everywhere,for critical points solve
$f_x = -2xy + 2x = 2x(1-y) = 0 ....y = 1 \text{ or } x = 0$
$f_y = 9y^2 - x^2 = 0......x^2 = 9y^2$
so all possible combinations are :
$(0,0)$,$(\pm 3, 1)$ ......3 critical points
$f(0,0) = 0$ $f(\pm 3, 1) = 3$

$f_{xx} = 2 - 2y$......A $\quad$ $f_{xy} = -2x$..........B $\quad$ $f_{yy} = 18y$..........C
the discriminant $D = B^2 - AC$
at $(\pm 3, 1)$ ....$A = 0, B = \mp 6, C = 18$ so $D = 36 > 0$.......*saddle* points
at $(0,0)$.....$D = 0$..... NO conclusion so we have to go back to the values of f
$f(0,y) = 3y^3$ $\quad$ values are positive for $y > 0$ and negative for $y < 0$.
.......*saddle* point again.

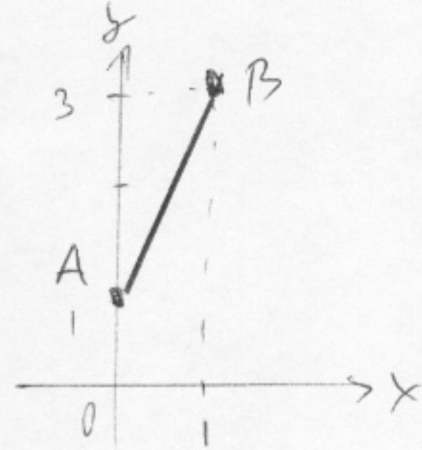
A

1a)

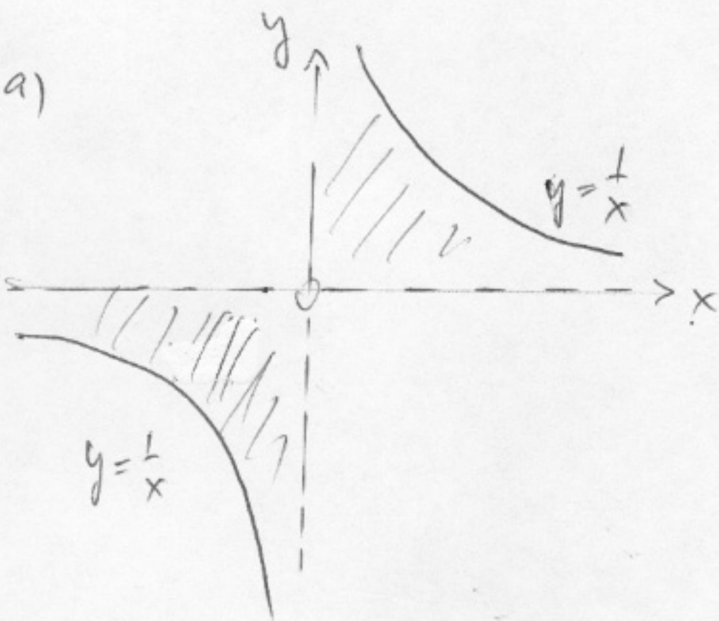


A

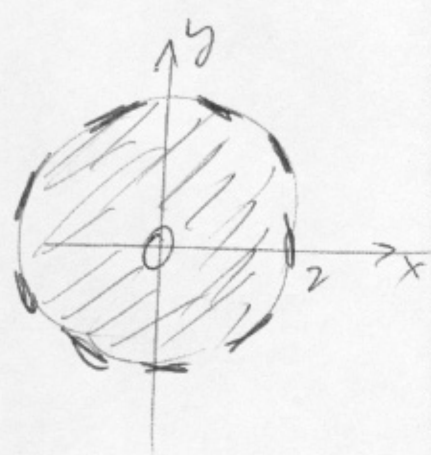
1b)



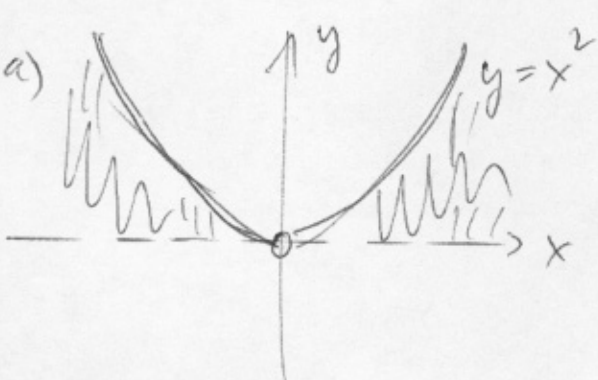
B 1a)



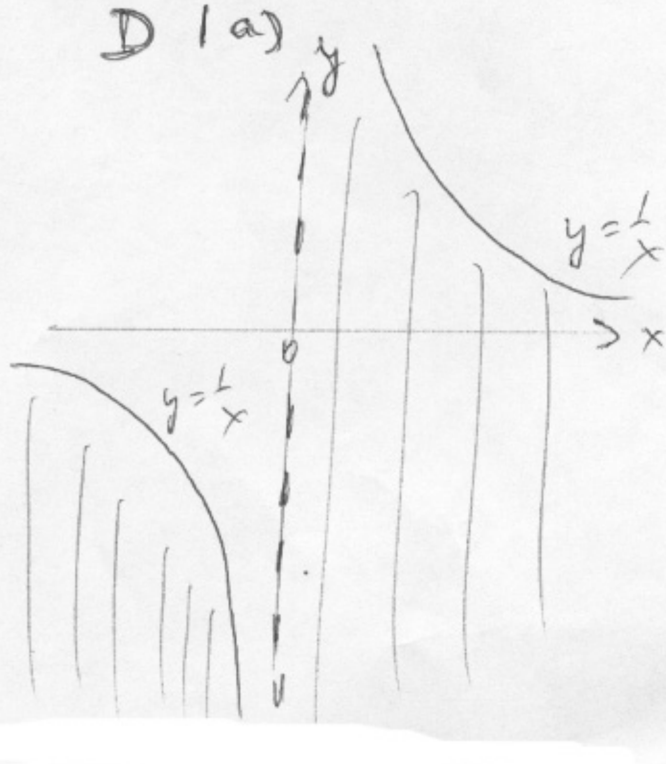
B 1b)



C 1a)



D 1a)



C 2b)