The University of Calgary
Department of Mathematics and Statistics
MATH 353
FINAL HANDOUT-SOLUTION

1. Sketch the region of integration and evaluate
$$\int_{-1}^{0}\left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4+1}\right)dy.$$

For 1)

given $\quad -1 \le y \le 0 \qquad -1 \le x \le \sqrt[3]{y}$

change the order

$$-1 \le x \le 0 \qquad x^3 \le y \le 0 \qquad \text{then}$$

$$\int_{-1}^{0}\left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4+1}\right)dy = \int_{-1}^{0}\left(\frac{1}{x^4+1}\int_{x^3}^{0} dy\right)dx =$$

$$= \int_{-1}^{0} \frac{-x^3dx}{x^4+1} = \frac{1}{4}\left[-\ln\left(x^4+1\right)\right]_{-1}^{0} = \frac{\ln 2}{4}.$$

2. Find all points on the sphere $x^2+y^2+z^2=36$ closest to $P\,(1,2,2)$.

For 2)

distance is $\sqrt{(x-1)^2+(y-2)^2+(z-2)^2}$ so looking for minimum of

$f(x,y,z) = (x-1)^2+(y-2)^2+(z-2)^2$ with $g(x,y,z) = x^2+y^2+z^2 = 36$

$\nabla f = \lambda \nabla g \qquad 2\,(x-1) = \lambda 2x \qquad 2\,(y-2) = \lambda 2y \qquad 2\,(z-2) = \lambda 2z$

gives $\lambda = \dfrac{x-1}{x} = \dfrac{y-2}{y} = \dfrac{z-2}{z}$ since $\lambda xyz \neq 0$

thus $z = y$ and $xy - y = yx - 2x \qquad y = 2x$ back to the sphere

$x^2+4x^2+4x^2 = 9x^2 = 36 \qquad x = \pm 2$ and we have 2 C.P.

$(-2,-4,-4)$ distance is 9 and

$(2,4,4)$ distance is 3 $\qquad$ minimum $\qquad$ the closest point

3. Express the integral $\displaystyle\iint_D \frac{x+y}{x^2+y^2}dxdy$

where D is the region above the line $x+y=2$ and inside the circle $x^2+y^2=4$

(a) as interated integrals in cartesian coordinates;

(b) as iterated integrals in polar coordinates then evaluate only once.

For 3 a)

sketch the set

$D = \{\ x + y \geq 2 \text{ and } x^2 + y^2 \leq 4\} = \{0 \leq x \leq 2, 2 - x \leq y \leq \sqrt{4 - x^2}\}$

so

$$\iint_D \frac{x+y}{x^2+y^2} dxdy = \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{x+y}{x^2+y^2} dy \right) dx = \text{OR} = \int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} \frac{x+y}{x^2+y^2} dx \right) dy$$

for b)

$D^* = \{r(\cos\theta + \sin\theta) \geq 2 \text{ and } 0 \leq r \leq 2, \theta \in \left[0, \frac{\pi}{2}\right]\}$

$$\iint_D \frac{x+y}{x^2+y^2} dxdy = \iint_{D^*} \frac{r(\cos\theta + \sin\theta)}{r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(\int_{\frac{2}{\cos\theta + \sin\theta}}^{2} dr \right) d\theta =$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(2 - \frac{2}{\cos\theta + \sin\theta}\right) d\theta = \int_0^{\frac{\pi}{2}} 2(\cos\theta + \sin\theta)\, d\theta - 2 \cdot \frac{\pi}{2} =$$

$$= 2\left[\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}} - \pi = 4 - \pi.$$

4. Evaluate $\displaystyle\iiint_B \frac{z dV}{\sqrt{x^2+y^2}}$

where $B = \{(x, y, z)\,; 1 \leq z \leq 4 \text{ and } z \geq x^2 + y^2\}$.

For 4)

sketch the set B a layer of a paraboloid

use cylindrical coordinates

$B^* = \{1 \leq z \leq 4 \text{ and } z \geq r^2, \theta \in [0, 2\pi]\}$

so $$\iiint_B \frac{z dV}{\sqrt{x^2+y^2}} = \iiint_{B^*} \frac{z}{\sqrt{r^2}} r dr dz d\theta = 2\pi \iint_{D^*} z dr dz$$

where $D^* = \{1 \leq z \leq 4\ , 0 \leq r \leq \sqrt{z}\}$

the integral $I = 2\pi \displaystyle\int_1^4 z\left(\int_0^{\sqrt{z}} dr\right) dz = 2\pi \int_1^4 z^{\frac{3}{2}} dz = \frac{4\pi}{5}\left[z^{\frac{5}{2}}\right]_1^4 = \frac{4\pi}{5}\left[2^5 - 1\right] = \frac{124\pi}{5}.$

5. Find the surface area of S

where S is the part of $z = \sqrt{3x^2 + 3y^2}$ below the plane $x + z = 4$.

For 5)

$SA = \displaystyle\iint_S dS = \iint_D \|\mathbf{n}\|\, dxdy$ we have to find D and $\mathbf{n} = (\nabla z, -1) = \left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}, \frac{\sqrt{3}y}{\sqrt{x^2+y^2}}, -1\right)$

and $\|\mathbf{n}\|^2 = \dfrac{3x^2+3y^2}{x^2+y^2} + 1 = 4 \qquad \|\mathbf{n}\| = 2$

for D find the intersection of the given cone and plane

$z = 4 - x = \sqrt{3x^2+3y^2} \qquad 16 - 8x + x^2 = 3x^2 + 3y^2$

$16 = 2x^2 + 8x + 3y^2 \qquad 24 = 2(x+2)^2 + 3y^2$

finally $\qquad D = \{2(x+2)^2 + 3y^2 \le 24\} = \{\left(\dfrac{x+2}{\sqrt{12}}\right)^2 + \left(\dfrac{y}{\sqrt{8}}\right)^2 \le 1\}$

so $\qquad SA = 2 \cdot area$ of an ellipse $= 2\pi ab = 2\pi\sqrt{12}\sqrt{8} = 8\pi\sqrt{6}.$

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6. Find $\displaystyle\oint_c \mathbf{F} \bullet \mathbf{ds}$

where $\mathbf{F} = (y^3 x + \cos(x^2), e^{y^2} + \sin(\pi x))$ and c is boundary of the triangle T from $(0,2)$ to $(2,2)$ to $(2,0)$ and back to $(0,2)$.

For 6)

the curve is closed and $n = 2$ use Green's theorem

$\displaystyle\oint_c \mathbf{F} \cdot \mathbf{ds} = -\iint_T \left[(F_2)_x - (F_1)_y\right] dxdy$ because of the orientation of c

where $T = \{0 \le x \le 2, 2 - x \le y \le 2\} = \{0 \le y \le 2, 2 - y \le x \le 2\}$

and $\qquad \left[(F_2)_x - (F_1)_y\right] = \pi\cos\pi x - 3xy^2$

so the integral $\displaystyle = -\iint_T [\pi\cos\pi x - 3xy^2]\, dxdy = \int_0^2 3y^2 \left(\int_{2-y}^{2} x dx\right) dy - \int_0^2 \pi\cos\pi x \left(\int_{2-x}^{2} dy\right) dx =$

$\displaystyle = \int_0^2 3y^2 \cdot \frac{2^2 - (2-y)^2}{2} dy - \int_0^2 x \cdot \pi\cos\pi x dx = \frac{3}{2}\int_0^2 (4y^3 - y^4)\, dy - [x\sin\pi x]_0^2 - \left[\frac{\cos\pi x}{\pi}\right]_0^2 =$

$\displaystyle = \frac{3}{2}\left[2^4 - \frac{2^5}{5}\right] - 0 = \frac{72}{5}.$

7. Show that for any smooth vector field $\mathbf{F}(x,y)$ and any smooth real-velued function $\phi(x,y)$

$div\,(\phi\mathbf{F}) = grad\phi \bullet \mathbf{F} + \phi div\ \mathbf{F} \qquad \nabla \cdot (\phi\mathbf{F}) = \nabla\boldsymbol{\phi} \cdot \mathbf{F} + \phi\,(\nabla \cdot \mathbf{F})$.

For 7)

$div(\phi\mathbf{F}) = (\phi F_1)_x + (\phi F_2)_y = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 =$

$= \phi(F_1)_x + \phi(F_2)_y + (\phi_x, \phi_y) \bullet (F_1, F_2) = \phi div\mathbf{F} + grad\phi \bullet \mathbf{F}$

8. Evaluate $\displaystyle\oint_c \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F} = (x^2 + y, y^3 - x, z^4)$

and c is given as $\{x^2 + y^2 = 4\} \cap \{2x - 3y + z = 2\}$ oriented positively.

.For 8)

since c is closed we can use Stokes's Theorem

$$I = \oint_c \mathbf{F} \cdot \mathbf{ds} = \iint_S curl\ \mathbf{F} \cdot \mathbf{dS}$$

where

$$curl\ \mathbf{F} = \left\| \begin{array}{ccc} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2+y & y^3-x & z^4 \end{array} \right\| = (0,0,-2)$$

and

S is a part of the plane $z = 2-2x+3y$ so $\mathbf{n} = (-\nabla z, 1) = (2,-3,1)$

inside the cylinder $\{x^2+y^2 \leq 4\} = D$

thus

$$I = \iint_D (-2)\, dxdy = -2\cdot\text{area of } D = -8\pi.$$

9. Find the flux of $\mathbf{F} = (x^2, y^2, z^2)$ outward

   from the closed surface $S = \{x^2+y^2+4(z-1)^2 = 4\}$.

   For 9)

   since S is closed use Divergence theorem

   $div\mathbf{F} = 2x+2y+2z$ and

   $$I = \iint_S \mathbf{F} \cdot \mathbf{dS} = 2\iiint_B (x+y+z)\, dxdydz \text{ where } B = \{x^2+y^2+4(z-1)^2 \leq 4\}$$

   ellipsoid symmetrical in x and in y so $I = 0+0+2\iiint_B zdxdydz =$

   $$= 2\int_0^2 z \iint_{D_z} dxdy$$

   where $D_z = \{x^2+y^2 \leq 4-4(z-1)^2 = 8z-4z^2\}$ circle for each fixed z

   thus $$I = 2\int_0^2 z\pi\left[8z-4z^2\right] dz = 8\pi \int_0^2 (2z^2-z^3)dz = 8\pi\left[\frac{2}{3}\cdot 2^3 - \frac{2^4}{4}\right] = \frac{32}{3}\pi$$

   Or cylindrical coordinates $2\iiint_B zdxdydz = 4\pi \iint_{D^*} zrdrdz$

   where $D^* = \{r^2 + 4(z-1)^2 \leq 4\}$

   the integral $= \int_0^2 z \left(\int_0^{\sqrt{4-4(z-1)^2}} rdr \right) dz = 2\int_0^2 z\,(2z-z^2)\, dz = ...$

   Or modified spherical coord.

10. Evaluate $\int_c \mathbf{F} \bullet \mathbf{ds}$ where $\mathbf{F} = (y, x, 2z)$ and

$c = \{z = 2xy\} \cap \{x^2 + y^2 = 2\}$ from $A(-1, 1, -2)$ to $B(1, 1, 2)$.

For 10)

the curve is not closed but the field is conservative

so we can find a potential

$f(x, y, z) = xy + z^2$ and $\int_c \mathbf{F} \bullet \mathbf{ds} = f(B) - f(A) = 5 - 3 = 2$

OR find a parametrization of c

from the cylinder $x = \sqrt{2}\cos t,\ y = \sqrt{2}\sin t$ and from $z = 2xy$

we have $\mathbf{r}(t) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 4\cos t\sin t\right) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 2\sin 2t\right)$

now for A $t = \frac{3}{4}\pi$, and for B $t = \frac{\pi}{4}$

$\mathbf{r}'(t) = \left(-\sqrt{2}\sin t, \sqrt{2}\cos t, 4\cos 2t\right)$ and the field on c

$\mathbf{F} = (y, x, 2z)_c = \left(\sqrt{2}\sin t, \sqrt{2}\cos t, 4\sin 2t\right)$ then

$\mathbf{F} \cdot \mathbf{r}' = -2\sin^2 t + 2\cos^2 t + 8\sin 4t$

$$\int_c \mathbf{F} \cdot \mathbf{ds} = \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \mathbf{F} \cdot \mathbf{r}' dt = -\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} [2\cos 2t + 8\sin 4t]\, dt =$$

$$= [-\sin 2t + 2\cos 4t]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 1 + 1 - 2 + 2 = 2.$$

11. For the vector field $\mathbf{F}(x, y, z) = (\arctan z(x^2 + y^2), \ln(1 + y^2 + z^2), y\ e^{xyz})$ calculate $div\mathbf{F}$ and $curl\mathbf{F}$ in the domain.

For 11)

for any (x, y, z)

$$div\mathbf{F} = \frac{2xz}{1 + z^2(x^2 + y^2)^2} + \frac{2y}{1 + y^2 + z^2} + xy^2\ e^{xyz}$$

and

$$curl\mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arctan z(x^2 + y^2) & \ln(1 + y^2 + z^2) & ye^{xyz} \end{bmatrix} =$$

$$= \left(e^{xyz}(1 + xyz) - \frac{2z}{1 + y^2 + z^2}, -y^2 z e^{xyz} + \frac{x^2 + y^2}{1 + z^2(x^2 + y^2)^2}, -\frac{2yz}{1 + z^2(x^2 + y^2)^2}\right).$$