MATH 353
Handout #5

1. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies inside the paraboloid $z = x^2 + y^2$.
2. Find the flux of $\mathbf{F} = \mathbf{i}+\mathbf{j}+z(x^2+y^2)^2\mathbf{k}$ out of the surface (including top and bottom) $S = \{(x, y, z)\,; x^2 + y^2 = 4, 0 \le z \le 3\}$.
3. Find the surface area of S
   - (a) which is the part of the cylinder $x^2 + y^2 = 4$ in the first octant below the plane $2x + y + z = 5$;
   - (b) which is the part of the plane $2x + y + z = 5$ inside the cylinder $x^2 + y^2 = 4$.
4. Find the flux $\iint\limits_S \mathbf{F}\cdot\mathbf{dS}$ where S is the part of the cylinder $y^2 + z^2 = 4$ which lies inside the cylinder $x^2 + y^2 = 4$,above the xy-plane,oriented upward, and the field is $\mathbf{F}(x, y, z) = (x^2yz, y, xz)$.
5. Evaluate $\int\limits_S\int zx\; dS$ where S is the part of $z = \frac{x^2}{2}$ which lies inside $x^2 + y^2 = 1, x > 0, y < 0$.
6. Evaluate $\iint\limits_S x^2 dS$ where S is the part of the plane $x + y + z = 1$ inside the cyllinder $x^2 + 2y^2 = 1$.

Solutions.

For1)

let's find the intersection of two given surfaces
$z = x^2 + y^2 \rightarrow z + z^2 = 2$ so $z^2 + z - 2 = 0$ has positive sol. $z = 1$
and we can describe the surface
$S = \left\{z = \sqrt{2 - x^2 - y^2}, (x, y) \in D\right\}$where $D = \{x^2 + y^2 \le 1\}$
and normal $\mathbf{n} = \pm(\nabla z, -1) = \pm\left(\frac{x}{\sqrt{2-x^2-y^2}}, \frac{y}{\sqrt{2-x^2-y^2}}, 1\right)$
and $\|\mathbf{n}\| = \sqrt{\frac{2}{2-x^2-y^2}}$,finally the surface area

$$A = \iint\limits_S dS = \iint\limits_D \|\mathbf{n}\|\, dxdy = \sqrt{2}\iint\limits_D \frac{dxdy}{\sqrt{2-x^2-y^2}} = (\text{ polar})$$

$$= 2\pi\sqrt{2}\int_0^1 \frac{rdr}{\sqrt{2-r^2}} = 2\sqrt{2}\pi\left[-\sqrt{2-r^2}\right]_0^1 = 2\pi\left[2-\sqrt{2}\right].$$

For 2)
the field on the lateral part of S_l is $\mathbf{F} = (1, 1, 16z)$
but since this part is vertical

we cannot describe it as $z = f(x, y)$ so parametrize $\mathbf{r}(u, v)$:

$x = 2\cos u \quad y = 2\sin u \quad z = v \quad u \in [0, 2\pi], v \in [0, 3]$

$\frac{\partial \mathbf{r}}{\partial u} = (-2\sin u, 2\cos u, 0) \qquad \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)$ and the normal vector

$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (2\cos u, 2\sin u, 0) \ (\|\mathbf{n}\| = 2)$

$\mathbf{F} \cdot \mathbf{n} = 2(\cos u + \sin u)$

$\iint\limits_{S_l} \mathbf{F} \cdot \mathbf{dS} = 2 \int\limits_0^{2\pi} \int\limits_0^3 (\cos u + \sin u)\, dv du = 0$ because of the periodicity.

The top could be described as $\quad z = 3$

$(x, y) \in D = \{x^2 + y^2 \le 4\}$ $\mathbf{n} = (0, 0, 1)$

the field on the top is $\mathbf{F} = (1, 1, 3(x^2 + y^2)^2)$ and $\mathbf{F} \cdot \mathbf{n} = 3(x^2 + y^2)^2$

the flux comming out from the top is

$$\iint\limits_{top} \mathbf{F} \cdot \mathbf{dS} = \iint\limits_D 3(x^2 + y^2)^2 dx dy = (\text{ polar}) = 6\pi \int\limits_0^2 r^5 dr = 2^6\pi.$$

The bottom could be described as $\quad z = 0$

$(x, y) \in D = \{x^2 + y^2 \le 4\}$ $\mathbf{n} = (0, 0, -1)$

the field on the bottom is $\mathbf{F} = (1, 1, 0)$ and $\mathbf{F} \cdot \mathbf{n} = 0.$

Therefore the only contribution to the flux is out from the top and

$\iint\limits_S \mathbf{F} \cdot \mathbf{dS} = 64\pi.$

For 3 a)

the surface S is vertical so parametrize:

$x = 2\cos u, y = 2\sin u,\ u \in \left[0, \frac{\pi}{2}\right]$, and $0 \le z \le 5 - 2x - y$ (below the plane)

$\mathbf{r}(u, v) = (2\cos u, 2\sin u, v)$ and $D = \left\{u \in \left[0, \frac{\pi}{2}\right], 0 \le v \le 5 - 4\cos u - 2\sin u\right\}$

$$\frac{\partial \mathbf{r}}{\partial u} = (-2\sin u, 2\cos u, 0), \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1) \text{ so } \mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (2\cos u, 2\sin u, 0)$$

thus $\|\mathbf{n}\| = 2$ and $SA = \iint\limits_S dS = \iint\limits_D \|\mathbf{n}\|\, du dv = 2 \int\limits_0^{\frac{\pi}{2}} \left(\int\limits_0^{5-4\cos u - 2\sin u} dv \right) du =$

$$= 2 \int\limits_0^{\frac{\pi}{2}} (5 - 4\cos u - 2\sin u) du = 5\pi + [-8\sin u + 4\cos u]_0^{\frac{\pi}{2}} = 5\pi - 8 - 4 = 5\pi - 12.$$

For 3 b)

this time $S \qquad z = 5 - 2x - y$ and $(x, y) \in D = \{x^2 + y^2 \le 4\}$

then $\mathbf{n} = (-2, -1, -1)$ and $\|\mathbf{n}\| = \sqrt{6}$

7. $SA = \iint\limits_S dS = \iint\limits_D \|\mathbf{n}\|\, dx dy = \sqrt{6} \cdot \text{area of } D = 4\sqrt{6}\pi.$

For 4)

$S : z = \sqrt{4 - y^2}, (x, y) \in D = \{x^2 + y^2 \le 4\}$ since $z > 0$

normal vector $\mathbf{n} = -(\nabla z, -1) = \left(0, \frac{y}{\sqrt{4 - y^2}}, 1\right)$

since upward means positive z-coordinate

$\mathbf{F} = (x^2 yz, y, xz)$ on $S \qquad \mathbf{F} = \left(..., y, x\sqrt{4 - y^2}\right)$ and

8. $$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \mathbf{F}\cdot\mathbf{n}\,dxdy = \iint\limits_D \left[\frac{y^2}{\sqrt{4-y^2}} + x\sqrt{4-y^2}\right] dxdy =$$

$$= \int\limits_{-2}^{2} \Big(\frac{y^2}{\sqrt{4-y^2}} \int\limits_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\Big)dy + \int\limits_{-2}^{2} \sqrt{4-y^2}\left(\int\limits_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xdx\right)dy =$$

$$= 2\int\limits_{0}^{2} \frac{y^2}{\sqrt{4-y^2}}\cdot 2\sqrt{4-y^2}dy + 0(x \text{ is an odd f}) = 4\int\limits_{0}^{2} y^2dy = \tfrac{4}{3}\cdot 8 = \frac{32}{3}.$$

For 5)

$S : z = \frac{x^2}{2}, (x,y) \in D = \{x^2+y^2 \le 1, x > 0, y < 0\}$

we can slice $D = \{0 < x \le 1, -\sqrt{1-x^2} < y < 0\}$

normal vector $\mathbf{n} = (\nabla z, -1) = (x, 0, -1)$ and $\|\mathbf{n}\| = \sqrt{x^2+1}$

and

$$\int\limits_S\int zx\ dS\ = \int\limits_D\int \tfrac{x^2}{2}\cdot x\sqrt{x^2+1}\ dxdy = \tfrac{1}{2}\int\limits_0^1 x^3\sqrt{x^2+1}\Big(\int_{-\sqrt{1-x^2}}^{0} dy\Big)dx\ =$$

$$= \tfrac{1}{2}\int\limits_0^1 x^3\sqrt{1-x^4}dx = (u = 1-x^4, du = -4x^3dx) = \tfrac{1}{8}\int\limits_0^1 \sqrt{u}du = \tfrac{1}{8}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \tfrac{1}{12}.$$

For 6)

Evaluate where S : $\qquad z = 1 - x - y$ for $(x,y) \in D = \{x^2 + 2y^2 \le 1\}$

so $\mathbf{n} = (\nabla z, -1) = (-1,-1,-1)$ and $\|\mathbf{n}\| = \sqrt{3}$

and $\iint\limits_S x^2dS = \int\limits_D\int x^2\sqrt{3}\ dxdy$ (modified polar . or cartesian coord.)

$$x = r\cos\theta \qquad y = \frac{1}{\sqrt{2}}r\sin\theta \qquad dxdy = \frac{1}{\sqrt{2}}rdrd\theta \qquad x^2+2y^2 = r^2$$

then $D^* = \{0 \le r \le 1, 0 \le \theta \le 2\pi\}$ and the integral

$$= \sqrt{\frac{3}{2}}\int\limits_0^{2\pi}\int\limits_0^{1} r^2\cos^2\theta r d\theta dr = \sqrt{\frac{3}{2}}\left[\frac{r^4}{4}\right]_0^1 \int\limits_0^{2\pi} \frac{1+\cos 2\theta}{2}d\theta = \frac{\sqrt{3}\pi}{4\sqrt{2}}$$

OR $\qquad x \in [-1,1]$ and $y \in \left[-\frac{1}{\sqrt{2}}\sqrt{1-x^2}, \frac{1}{\sqrt{2}}\sqrt{1-x^2}\right]$

$$\iint\limits_S x^2dS = \int\limits_D\int x^2\sqrt{3}\ dxdy = \sqrt{\frac{3}{2}}\int\limits_{-1}^{1} x^2\left(2\sqrt{1-x^2}\right)dx \text{ Table..}$$