

**MATH 353**  
**Handout #2 Solutions**

1. Find absolute extrema of  $f(x, y) = \frac{1}{8}x^3 + y^3$  on the circle  $x^2 + y^2 = 65$

**SOLUTION For 1)**

The function is cont. and the set is closed and bounded so use Lagrange multiplier method where  $g(x, y) = x^2 + y^2 = 65$

Solve  $\nabla f = \lambda \nabla g$

$$\frac{3}{8}x^2 = \lambda 2x \rightarrow \rightarrow \rightarrow x = 0 \text{ or } \frac{3}{8}x = 2\lambda$$

$$3y^2 = \lambda 2y \rightarrow \rightarrow \rightarrow y = 0 \text{ or } 3y = 2\lambda$$

For  $x = 0$  back to the circle  $y = \pm\sqrt{65}$ , similarly for  $y = 0, x = \pm\sqrt{65}$

for  $xy \neq 0$

$$2\lambda = \frac{3}{8}x = 3y \implies x = 8y \text{ back to the circle } 64y^2 + y^2 = 65$$

$$y = \pm 1 \text{ and } x = \pm 8$$

6 critical points:  $(0, \pm\sqrt{65}), (\pm\sqrt{65}, 0), (\pm 8, \pm 1)$

values of  $f$ : .....  $\pm 65\sqrt{65}$ .....  $\pm \frac{65}{8}\sqrt{65}$ .....  $\pm 65$

So **max** is  $65\sqrt{65}$  at  $(0, \sqrt{65})$  and **min** is  $-65\sqrt{65}$  at  $(0, -\sqrt{65})$

2. Find the absolute extrema of  $f(x, y) = x^2 + y^2$  on the surface  $S = \{\frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0\}$ .

**SOLUTION For 2)**

Notice that the roles of  $f$  and  $g$  are interchanged, and we need  $x, y \geq 0$  to make  $S$  bounded.

$$\text{so } g(x, y) = \frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0$$

Solve  $\nabla f = \lambda \nabla g$

$$2x = \lambda \frac{3}{8}x^2 \rightarrow \rightarrow \rightarrow x = 0 \text{ or } \frac{16}{x} = 3\lambda$$

$$2y = \lambda 3y^2 \rightarrow \rightarrow \rightarrow \rightarrow y = 0 \text{ or } \frac{2}{y} = 3\lambda$$

For  $x = 0$  back to  $S$   $y = \sqrt[3]{65} = 4.02$ , similarly, for  $y = 0, x = 2\sqrt[3]{65}$

for  $xy \neq 0$

$$3\lambda = \frac{16}{x} = \frac{2}{y} \text{ so } x = 8y \text{ back to } S \quad 64y^3 + y^3 = 65$$

$$y = 1, x = 8$$

3 critical points:  $(0, \sqrt[3]{65}) \quad (2\sqrt[3]{65}, 0) \quad (8, 1)$

values of  $f$   $65^{\frac{2}{3}} = 16.1 \quad 4 \cdot 65^{\frac{2}{3}} = 64.66 \quad 65$

So **max** is 65 at  $(8, 1)$  and **min** is  $65^{\frac{2}{3}}$  at  $(0, \sqrt[3]{65})$ .

3. Find absolute maximum and minima of  $f(x, y) = 2y^2 - x + x^2$  inside and on the triangle  $T$  with vertices  $O(0, 0)$ ,  $A(1, 1)$ ,  $B(1, -1)$ .

**SOLUTION For 3)**

Since the function is continuous and the set is closed and bounded we have to find all critical points inside and on the boundary, and check the values at those.

For critical points inside solve  $\nabla f = \mathbf{0}$

$$f_x = 2x - 1 = 0, f_y = 4y = 0 \text{ so } x = \frac{1}{2}, y = 0$$

Now, all critical points on the boundary

the boundary of  $T$  consists of 3 line segments

$$B_1 = \{y = x, 0 \leq x \leq 1\} \text{ and } B_2 = \{y = -x, 0 \leq x \leq 1\}$$

$$f(x, \pm x) = 3x^2 - x = h(x), h'(x) = 6x - 1 = 0 \text{ for } x = \frac{1}{6} \rightarrow y = \pm \frac{1}{6}$$

and the ends (corners)

$$B_3 = \{x = 1, -1 \leq y \leq 1\} \text{ and } f \text{ on } B_3 \text{ is } f(y, 1) = g(y) = 2y^2$$

$$\text{and } g'(y) = 4y = 0 \text{ for } y = 0, x = 1$$

Together all critical points in  $T$  and on  $\partial T$  are

$$\left(\frac{1}{2}, 0\right), \left(\frac{1}{6}, \pm \frac{1}{6}\right), (0, 0), (1, \pm 1), (1, 0)$$

Check the values of  $f$

$$\dots \frac{-1}{4} \dots \frac{-1}{12} \dots 0 \dots 2 \dots 0$$

So abs. max. value is 2 at the points  $(1, \pm 1)$

and abs. min. value is  $\frac{-1}{4}$  at the point  $\left(\frac{1}{2}, 0\right)$ .

4. Find the point on the plane  $x - 2y - z = 3$  closest to the point  $P(1, -1, 2)$ .

Justify!

**SOLUTION For 4)**

We are looking for minimum of the distance to  $P$  or the square of distance

$$f(x, y, z) = (x - 1)^2 + (y + 1)^2 + (z - 2)^2 \text{ and the constraint } g(x, y, z) = x - 2y - z = 3$$

Solve  $\nabla f = \lambda \nabla g$

$$2(x - 1) = \lambda$$

$$2(y + 1) = -2\lambda \dots \dots \dots \lambda = 2x - 2 = -y - 1 = 4 - 2z$$

$$2(z - 2) = -\lambda \dots \dots \dots 1 - 2x = y \dots \dots \dots z = 3 - x$$

back to the plane

$$x - 2(1 - 2x) - (3 - x) = 6x - 5 = 3 \rightarrow \rightarrow \rightarrow x = \frac{4}{3}, y = \frac{-5}{3}, z = \frac{5}{3}$$

and the point is  $\left(\frac{4}{3}, \frac{-5}{3}, \frac{5}{3}\right)$

To justify that we have found minimum

the set is not bounded but if we take a bounded part we have to have maximum and minimum

but the maximum will be on the boundary since the distance is increasing when we move far away therefore the critical point must be minimum.

5. Find absolute maximum of  $f(x, y, z) = xyz$

largest box with sides  $x, y, z$  on  $\{(x, y, z); 2xy + 2xz + 3yz = 144, x \geq 0, y \geq 0, z \geq 0\}$ .

(You may assume that there is an absolute maximum)

**Solutions.For 5)**

$\nabla f = \lambda \nabla g$  where  $g(x, y, z) = 2xy + 2xz + 3yz = 144$

$yz = \lambda(2y + 2z)$  if  $y = 0$  then  $z = 0 \dots$  but  $0 = 144$

so we can assume that all  $xyz \neq 0$

$xz = \lambda(2x + 3z)$   $\frac{1}{\lambda} = \frac{2y + 2z}{yz} = \frac{2x + 3z}{xz} = \frac{2x + 3y}{xy}$

$xy = \lambda(2x + 3y)$

from the first equality (cancel  $z$ )  $\dots 2yx + 2xz = 2xy + 3zy \rightarrow 2x = 3y$

from the second equality (cancel  $x$ )  $\dots 2xy + 3zy = 2xz + 3zy \rightarrow z = y$

back to the surface  $3z^2 + 3z^2 + 3z^2 = 9z^2 = 144 \rightarrow z^2 = \frac{144}{9} = 16 \rightarrow z = \pm 4$

and critical points are  $(\pm 6, \pm 4, \pm 4)$ , values of  $f$  are  $\pm 96$

so maximum is 96 at the point  $(6, 4, 4)$ .

to justify that this is a maximum

since  $z = \frac{144 - 2xy}{2x + 3y}$   $0 < xy < 72$   $y < \frac{72}{x}$

so  $z > \frac{\text{pos\#}}{2x + \frac{216}{x}}$  and as  $x \rightarrow \infty$  or  $x \rightarrow 0+$

$z \rightarrow 0$  and  $f \rightarrow 0$ .

6. (a) Evaluate  $\int_1^3 \left( \int_{-x}^{x^2} xe^{2y} dy \right) dx$ .

(b) Switch the order of integration in the integral above and sketch the region  $D$ .

**SOLUTION For 6a)**

$$\begin{aligned} I &= \int_1^3 \left( \int_{-x}^{x^2} xe^{2y} dy \right) dx = \int_1^3 x \left( \int_{-x}^{x^2} e^{2y} dy \right) dx = \int_1^3 x \left( \left[ \frac{e^{2y}}{2} \right]_{y=-x}^{y=x^2} \right) dx = \frac{1}{2} \int_1^3 x (e^{2x^2} - e^{-2x}) dx = \\ &= \frac{1}{2} \int_1^3 xe^{2x^2} dx \text{ (subst)} - \frac{1}{2} \int_1^3 xe^{-2x} dx \text{ (by parts)} = \end{aligned}$$

$$= \frac{1}{8} [e^{2x^2}]_1^3 - \frac{1}{2} \left[ x \frac{e^{-2x}}{-2} \right]_1^3 - \frac{1}{4} \int_1^3 e^{-2x} dx = \frac{1}{8} [e^{18} - e^2] + \frac{1}{4} [3e^{-6} - e^{-2}] + \frac{1}{8} [e^{-6} - e^{-2}] = ..$$

**For b)**

since we have three "left ends" we have to split the domain  $D = D_1 \cup D_2 \cup D_3$  where

$$D_1 = \{(x, y); -3 \leq y \leq -1, -y \leq x \leq 3\}, D_2 = \{(x, y); -1 < y \leq 1, 1 \leq x \leq 3\}$$

$$\text{and } D_3 = \{(x, y); 1 < y \leq 9, \sqrt{y} \leq x \leq 3\}$$

and the integral

$$I = \int_{-3}^{-1} \left( \int_{-y}^3 x e^{2y} dx \right) dy + \int_{-1}^1 \left( \int_1^3 x e^{2y} dx \right) dy + \int_{-1}^9 \left( \int_{\sqrt{y}}^3 x e^{2y} dx \right) dy.$$

7. Evaluate  $\iint_D \sqrt{2-x^2} dA$  where  $D$  is smaller region between  $y = x^2$  and  $x^2 + y^2 = 2$ .

and sketch the region

**SOLUTION For 7)**

find the intersection of the parabola  $y = x^2$  and the circle  $x^2 + y^2 = 2$

$$y^2 + y - 2 = 0 \quad (y-1)(y+2) = 0$$

but  $y$  must be positive so  $y = 1$  and  $x = \pm 1$  and  $D = \{-1 \leq x \leq 1, x^2 \leq y \leq \sqrt{2-x^2}\}$

$$\iint_D \sqrt{2-x^2} dA = \int_{-1}^1 \sqrt{2-x^2} \left( \int_{x^2}^{\sqrt{2-x^2}} dy \right) dx = \int_{-1}^1 \sqrt{2-x^2} (\sqrt{2-x^2} - x^2) dx =$$

$$= \int_{-1}^1 (2 - x^2 - x^2 \sqrt{2-x^2}) dx = (\text{even f.}) = 2 \int_0^1 \dots dx =$$

$$= 2 \cdot 2 - 2 \left[ \frac{x^3}{3} \right]_0^1 - 2 \left( \text{Table } a = \sqrt{2} \right) \left[ \frac{x}{8} (2x^2 - 2) \sqrt{2-x^2} + \frac{1}{2} \arcsin \frac{x}{\sqrt{2}} \right]_0^1 =$$

$$= 4 - \frac{2}{3} + 0 - \frac{1}{2} \arcsin \frac{1}{\sqrt{2}} = \frac{10}{3} - \frac{\pi}{4}.$$

8. Switch the order of integration in the integral  $\int_0^{\frac{\pi}{4}} \left( \int_0^{\tan x} f(x, y) dy \right) dx$ .

**SOLUTION For 8)**

given  $0 \leq x \leq \frac{\pi}{4}$  and  $0 \leq y \leq \tan x$  sketch

$y = \tan x$  is equivalent to  $\arctan y = x$  for  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  and  $\tan \frac{\pi}{4} = 1$

so  $0 \leq y \leq 1$  and  $\arctan y \leq x \leq \frac{\pi}{4}$

and the integral

$$\int_0^{\frac{\pi}{4}} \left( \int_0^{\tan x} f(x, y) dy \right) dx = \int_0^1 \left( \int_{\arctan y}^{\frac{\pi}{4}} f(x, y) dx \right) dy.$$

9. For  $\iint_D \frac{1}{x^2 + y} dA$  where  $D$  is the region between the x-axis and  $y = 4 - x^2$  sketch the region  $D$  and set up BOTH iterated integrals and evaluate one of them. (Hint:  $\lim_{x \rightarrow 0^+} x \ln x = 0$ ).

**SOLUTION For 9)**

the region is above x-axis and below parabola  $y = 4 - x^2$

so  $-2 \leq x \leq 2$   $0 \leq y \leq 4 - x^2$  or  $0 \leq y \leq 4$   $-\sqrt{4-y} \leq x \leq \sqrt{4-y}$

and

$$\iint_D \frac{1}{x^2 + y} dA = \int_{-2}^2 \left( \int_0^{4-x^2} \frac{1}{x^2 + y} dy \right) dx = \int_0^4 \left( \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2 + y} dx \right) dy$$

evaluate the first ordered iterated integrals

$$\int_{-2}^2 \left( \int_0^{4-x^2} \frac{1}{x^2 + y} dy \right) dx = \int_{-2}^2 [\ln(x^2 + y)]_{y=0}^{y=4-x^2} dx = \int_{-2}^2 (\ln 4 - \ln x^2) dx = 4 \ln 4 - 2 \int_0^2 \ln x^2 dx =$$

$$= 4 \ln 4 - 4 \int_0^2 \ln x dx = 8 \ln 2 - 4 [x \ln x - x]_0^2 = 8. \text{ (Otherwise } \ln x^2 = 2 \ln |x| \text{!).}$$

The other way

$$\int_0^4 \left( \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2 + y} dx \right) dy = \int_0^4 \left( \left[ \frac{1}{\sqrt{y}} \arctan \frac{x}{\sqrt{y}} \right]_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} \right) dy = \dots \text{harder}$$

10. Calculate the volume of the solid below the surface  $z = e^{(y-1)^2}$  and above the triangle  $T$

with vertices  $A(-1, 0)$ ,  $B(0, 1)$ ,  $C(2, 0)$  with vertical sides.

**SOLUTION for 10)**

$$V = \iint_T e^{(y-1)^2} dx dy \quad \text{it is easier to slice the triangle horizontally}$$

$$0 \leq y \leq 1 \quad \text{line}_{AB} \leq x \leq \text{line}_{BC}$$

where line AB  $y = x + 1$  or  $x = y - 1$

lineBC  $y = 1 - \frac{1}{2}x$  or  $x = 2 - 2y$

$$\text{so } V = \int_0^1 \left( e^{(y-1)^2} \int_{y-1}^{2-2y} dx \right) dy = \int_0^1 e^{(y-1)^2} (3 - 3y) dy = -\frac{3}{2} \int_1^0 e^u du = \frac{3}{2} [e - 1]$$

by subst.  $u = (y - 1)^2$

