

MATH 353 FINAL HANDOUT-SOLUTION

For 1) $f(x, y) = x^2(y + 1)^3 + y^2$
 $f_x = 2x(y + 1)^3 = 0$ $f_y = 3x^2(y + 1)^2 + 2y = 0$

if $x = 0$ then $y = 0$; if $y = -1$ then No sol.

the only C.P. is $(0, 0)$, *noeSecond.der.test*

$f_{xx} = 2(y + 1)^3$ $A = 2$
 $f_{xy} = 6x(y + 1)^2$ $B = 0$
 $f_{yy} = 6x^2(y + 1) + 2$ $C = 2$ $D = -8$
 $D < 0, A > 0$ loc. minimum

For abs. min. we have to show that $f(x, y) \geq f(0, 0) = 0$ for all (x, y)

BUT $f(R, -2) = -R^2 + 4 < 0$ for big R

thus it is NOT abs. min.

For 2)

$\int_{-1}^0 \left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4 + 1} \right) dy$. given $-1 \leq y \leq 0$ $-1 \leq x \leq \sqrt[3]{y}$

change the order $-1 \leq x \leq 0$ $x^3 \leq y \leq 0$

then

$\int_{-1}^0 \left(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4 + 1} \right) dy = \int_{-1}^0 \left(\frac{1}{x^4 + 1} \int_{x^3}^0 dy \right) dx =$
 $= \int_{-1}^0 \frac{-x^3 dx}{x^4 + 1} = \frac{1}{4} [-\ln(x^4 + 1)]_{-1}^0 = \frac{\ln 2}{4}.$

For 3)

on the sphere $x^2 + y^2 + z^2 = 36$ closest to $P(1, 2, 2)$.

distance is $\sqrt{(x - 1)^2 + (y - 2)^2 + (z - 2)^2}$ so looking for minimum of

$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$ with $g(x, y, z) = x^2 + y^2 + z^2 = 36$

$\nabla f = \lambda \nabla g$ $2(x - 1) = \lambda 2x$ $2(y - 2) = \lambda 2y$ $2(z - 2) = \lambda 2z$

gives $\lambda = \frac{x - 1}{x} = \frac{y - 2}{y} = \frac{z - 2}{z}$ since $\lambda xyz \neq 0$

thus $z = y$ and $xy - y = yx - 2x$ $y = 2x$ back to the sphere

$x^2 + 4x^2 + 4x^2 = 9x^2 = 36$ $x = \pm 2$ and we have 2 C.P.

for $(-2, -4, -4)$ distance is $\sqrt{9 + 36 + 36} = 9$ and

$(2, 4, 4)$ is **the closest point** since distance is 3 so minimum

For 4 a)

$D = \{x + y \geq 2 \text{ and } x^2 + y^2 \leq 4\} = \{0 \leq x \leq 2, 2 - x \leq y \leq \sqrt{4 - x^2}\}$

so

$\iint_D \frac{x + y}{x^2 + y^2} dx dy = \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{x + y}{x^2 + y^2} dy \right) dx = \text{OR} = \int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} \frac{x + y}{x^2 + y^2} dx \right) dy$

for b)

$D^* = \{r(\cos\theta + \sin\theta) \geq 2 \text{ and } 0 \leq r \leq 2, \theta \in [0, \frac{\pi}{2}]\}$

$$\begin{aligned} \iint_D \frac{x+y}{x^2+y^2} dx dy &= \iint_{D^*} \frac{r(\cos\theta + \sin\theta)}{r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(\int_{\frac{2}{\cos\theta + \sin\theta}}^2 dr \right) d\theta = \\ &= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(2 - \frac{2}{\cos\theta + \sin\theta} \right) d\theta = \int_0^{\frac{\pi}{2}} 2(\cos\theta + \sin\theta) d\theta - 2 \cdot \frac{\pi}{2} = \\ &= 2[\sin\theta - \cos\theta]_0^{\frac{\pi}{2}} - \pi = 4 - \pi. \end{aligned}$$

For 5) the set B is a layer of a full paraboloid
use cylindrical coordinates

$$B^* = \{1 \leq z \leq 4 \text{ and } z \geq r^2, \theta \in [0, 2\pi]\}$$

$$\text{so } \iiint_B \frac{z dV}{\sqrt{x^2+y^2}} = \iiint_{B^*} \frac{z}{\sqrt{r^2}} r dr dz d\theta = 2\pi \iint_{D^*} z dr dz$$

$$\text{where } D^* = \{1 \leq z \leq 4, 0 \leq r \leq \sqrt{z}\}$$

$$\text{the integral } I = 2\pi \int_1^4 z \left(\int_0^{\sqrt{z}} dr \right) dz = 2\pi \int_1^4 z^{\frac{3}{2}} dz = \frac{4\pi}{5} [z^{\frac{5}{2}}]_1^4 = \frac{4\pi}{5} [2^5 - 1] = \frac{124\pi}{5}.$$

$$\text{For 6)} \quad S = \{z = \sqrt{3x^2 + 3y^2} \quad x + z \leq 4\}$$

$$SA = \iint_S dS = \iint_D \|\mathbf{n}\| dx dy$$

$$\mathbf{n} = (\nabla z, -1) = \left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}, \frac{\sqrt{3}y}{\sqrt{x^2+y^2}}, -1 \right)$$

$$\text{and } \|\mathbf{n}\|^2 = \frac{3x^2 + 3y^2}{x^2 + y^2} + 1 = 4 \quad \|\mathbf{n}\| = 2$$

we have to find D the intersection of the given cone and plane

$$z = 4 - x = \sqrt{3x^2 + 3y^2} \quad 16 - 8x + x^2 = 3x^2 + 3y^2$$

$$16 = 2x^2 + 8x + 3y^2 \quad 24 = 2(x+2)^2 + 3y^2$$

$$\text{finally } D = \{2(x+2)^2 + 3y^2 \leq 24\} = \left\{ \left(\frac{x+2}{\sqrt{12}} \right)^2 + \left(\frac{y}{\sqrt{8}} \right)^2 \leq 1 \right\}$$

$$\text{so } SA = 2 \cdot \text{area of an ellipse} = 2\pi ab = 2\pi \sqrt{12} \sqrt{8} = 8\pi \sqrt{6}.$$

For 7)

the curve is closed, oriented clockwise, $n = 2$ use Green's theorem

$$\oint_c \mathbf{F} \cdot d\mathbf{s} = - \iint_T [(F_2)_x - (F_1)_y] dx dy \text{ because of the orientation of } c$$

$$\text{where } T = \{0 \leq x \leq 2, 2-x \leq y \leq 2\} = \{0 \leq y \leq 2, 2-y \leq x \leq 2\}$$

$$\text{and } [(F_2)_x - (F_1)_y] = \pi \cos \pi x - 3xy^2$$

$$= - \iint_T [\pi \cos \pi x - 3xy^2] dx dy = \int_0^2 3y^2 \left(\int_{2-y}^2 x dx \right) dy - \int_0^2 \pi \cos \pi x \left(\int_{2-x}^2 dy \right) dx =$$

$$= \int_0^2 3y^2 \cdot \frac{2^2 - (2-y)^2}{2} dy - \int_0^2 x \cdot \pi \cos \pi x dx = \frac{3}{2} \int_0^2 (4y^3 - y^4) dy - [x \sin \pi x]_0^2 - \left[\frac{\cos \pi x}{\pi} \right]_0^2 =$$

$$= \frac{3}{2} \left[2^4 - \frac{2^5}{5} \right] - 0 = \frac{72}{5}.$$

For 8)

$$\text{div}(\phi \mathbf{F}) = (\phi F_1)_x + (\phi F_2)_y = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 =$$

$$= \phi(F_1)_x + \phi(F_2)_y + (\phi_x, \phi_y) \bullet (F_1, F_2) = \phi \operatorname{div} \mathbf{F} + \operatorname{grad} \phi \bullet \mathbf{F}$$

For 9)

since c is closed, $n = 3$, we can use Stokes's Theorem

$$I = \oint_c \mathbf{F} \cdot d\mathbf{s} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

where

$$\operatorname{curl} \mathbf{F} = \left\| \begin{array}{ccc} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2 + y & y^3 - x & z^4 \end{array} \right\| = (0, 0, -2)$$

and

S is a part of the plane $z = 2 - 2x + 3y$ so $\mathbf{n} = (-\nabla z, 1) = (2, -3, 1)$

inside the cylinder $\{x^2 + y^2 \leq 4\} = D$ $\operatorname{curl} \mathbf{F} \cdot \mathbf{n} = -2$

thus $I = \iint_D (-2) dx dy = -2 \cdot \text{area of } D = -8\pi$.

For 10) since S is closed use Divergence theorem

$\operatorname{div} \mathbf{F} = 2x + 2y + 2z$ and

$$I = \iiint_S \mathbf{F} \cdot d\mathbf{S} = 2 \iiint_B (x + y + z) dx dy dz \text{ where } B = \{x^2 + y^2 + 4(z-1)^2 \leq 4\}$$

ellipsoid symmetrical in x and in y so $I = 0 + 0 + 2 \iiint_B z dx dy dz =$

$$= 2 \int_0^2 z \iint_{D_z} dx dy$$

where $D_z = \{x^2 + y^2 \leq 4 - 4(z-1)^2 = 8z - 4z^2\}$ circular disc for each fixed z with area $= \pi(8z - 4z^2)$

$$\text{thus } I = 2 \int_0^2 z \pi [8z - 4z^2] dz = 8\pi \int_0^2 (2z^2 - z^3) dz = 8\pi \left[\frac{2}{3} \cdot 2^3 - \frac{2^4}{4} \right] = \frac{32}{3}\pi$$

OR cylindrical coordinates $2 \iiint_B z dx dy dz = 4\pi \iint_{D^*} z r dr dz$

where $D^* = \{r^2 + 4(z-1)^2 \leq 4\}$

$$\text{the integral} = \int_0^2 z \left(\int_0^{\sqrt{4-4(z-1)^2}} r dr \right) dz = 2 \int_0^2 z(2z - z^2) dz = \dots$$

Or modified spherical coord. $z = 1 + 2\rho \cos \phi$.

For 11) the curve is not closed but the field is conservative

so we can find a potential

$$: f_x = y \quad f_y = x \quad f_z = 2z$$

$$f(x, y, z) = xy + z^2 \text{ and } \int_c \mathbf{F} \bullet d\mathbf{s} = f(B) - f(A) = 5 - 3 = 2$$

OR find a parametrization of c

from the cylinder $x = \sqrt{2} \cos t$, $y = \sqrt{2} \sin t$ and from $z = 2xy$

we have $\mathbf{r}(t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 4 \cos t \sin t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 2 \sin 2t)$

now for A $t = \frac{3}{4}\pi$, and for B $t = \frac{\pi}{4}$

$\mathbf{r}'(t) = (-\sqrt{2} \sin t, \sqrt{2} \cos t, 4 \cos 2t)$ and the field on c

$\mathbf{F} = (y, x, 2z)_c = (\sqrt{2} \sin t, \sqrt{2} \cos t, 4 \sin 2t)$ then

$$\mathbf{F} \cdot \mathbf{r}' = -2 \sin^2 t + 2 \cos^2 t + 8 \sin 4t$$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \mathbf{F} \cdot \mathbf{r}' dt = - \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} [2 \cos 2t + 8 \sin 4t] dt =$$

$$= [-\sin 2t + 2 \cos 4t]_{\frac{3}{4}\pi}^{\frac{\pi}{4}} = 1 + 1 - 2 + 2 = 2.$$

For 12)

$$\mathbf{F}(x, y, z) = (\arctan z(x^2 + y^2), \ln(1 + y^2 + z^2), y e^{xyz})$$

for any (x, y, z)

$$\operatorname{div} \mathbf{F} = \frac{2xz}{1 + z^2(x^2 + y^2)^2} + \frac{2y}{1 + y^2 + z^2} + xy^2 e^{xyz}$$

$$\text{and } \operatorname{curl} \mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arctan z(x^2 + y^2) & \ln(1 + y^2 + z^2) & ye^{xyz} \end{bmatrix} =$$

$$= \left(e^{xyz} (1 + xyz) - \frac{2z}{1 + y^2 + z^2}, -y^2 z e^{xyz} + \frac{x^2 + y^2}{1 + z^2(x^2 + y^2)^2}, -\frac{2yz}{1 + z^2(x^2 + y^2)^2} \right).$$