

The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #4 Solution

1. Given $\mathbf{F}(x, y, z) = (3x^2yz, kyz + x^3z, x^3y + 1 + y^2)$.

(a) Find the value of k so that the field \mathbf{F} is conservative.

(b) Then, find a potential of \mathbf{F} .

For 1a)

necessary condition $(F_1)_y = 3x^2z = (F_2)_x = 3x^2z$

$(F_1)_z = 3x^2y = (F_3)_x = 3x^2y$, and finally

$(F_2)_z = ky + x^3 = (F_3)_y = x^3 + 2y$ gives us $k = 2$.

For 1b)

$f_x = F_1 = 3x^2yz$ so by integrating with respect to x :

$f(x, y, z) = \int (3x^2yz \, dx + c(y, z)) = x^3yz + c(y, z)$

differentiate $f_y = F_2 = 2yz + x^3z = x^3z + \frac{\partial c}{\partial y}$ thus $\frac{\partial c}{\partial y} = 2yz$ and $c(y, z) = y^2z + c(z)$

together $f(x, y, z) = x^3yz + y^2z + c(z)$

differentiate $f_z = F_3 = x^3y + 1 + y^2 = x^3y + y^2 + c'(z)$ thus $c' = 1$ and $c(z) = z + c$

and finally the general potential $f(x, y, z) = x^3yz + y^2z + z + c$ where c is any constant.

2. Evaluate $\int_c f \, ds$ where $f(x, y, z) = z^2$ and c is the part of the line of intersection of two planes;

$x + y - z = 1$ and $2x + y - 3z = 0$ between the xy -plane and the point $D(3, 0, 2)$

For 2)

first let's find the line, a direction vector $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$ (of normal vectors of the given planes)

$\mathbf{d} = (1, 1, -1) \times (2, 1, -3) = (-2, 1, -1)$ so

$(x, y, z) = (3, 0, 2) + t(-2, 1, -1)$ and when $t = 0$ we get D

$x = 3 - 2t, y = t, z = 2 - t$

to get a point on the xy -plane $z = 0$ so $t = 2$ and the point is $P(-1, 2, 0)$

Together $\mathbf{r}(t) = (3 - 2t, t, 2 - t)$ $t \in [0, 2]$, $\mathbf{r}'(t) = (-2, 1, -1) = \mathbf{d}$

the given function f evaluated on c $f \circ \mathbf{r} = (2 - t)^2$

$$\int_c f \, ds = \int_0^2 (2 - t)^2 \|\mathbf{r}'(t)\| \, dt = \sqrt{6} \left[\frac{(t - 2)^3}{3} \right]_0^2 = \frac{8}{3}\sqrt{6}.$$

3. For $\mathbf{F}(x, y) = (ky^2 + x, xy - \frac{1}{\sqrt{y}})$ find the value for k

so that the field is conservative, then find a potential.

For 3)

$$F_1 = ky^2 + x \text{ and } F_2 = xy - \frac{1}{\sqrt{y}} \text{ for } y > 0$$

$$(F_1)_y = 2ky = (F_2)_x = y \text{ so } k = \frac{1}{2}$$

then

$$f_x = \frac{1}{2}y^2 + x \text{ and } f_y = xy - \frac{1}{\sqrt{y}}$$

$$f = \int f_x dx = \int \left(\frac{1}{2}y^2 + x\right) dx + c(y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 + c(y)$$

$$f_y = xy + c'(y) = xy - \frac{1}{\sqrt{y}} \quad c'(y) = -\frac{1}{\sqrt{y}} \text{ for } y > 0$$

$$\text{and } c(y) = -2\sqrt{y} \quad \text{together } f(x, y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 - 2\sqrt{y} + \text{const.}$$

4. Evaluate $\int_c z ds$ and c is the intersection of the plane $z - y = 1$ and the paraboloid $0 = x - y^2$ between $A(1, -1, 0)$ and $B(0, 0, 1)$.

For 4)

intersection of $z - y = 1$ and $0 = x - y^2$ between $A(1, -1, 0)$ and $B(0, 0, 1)$

$$y = t \text{ and } \mathbf{r}(t) = (t^2, t, 1 + t) \text{ for } t \in [-1, 0]$$

$$\mathbf{r}'(t) = (2t, 1, 1) \text{ and } \|\mathbf{r}'(t)\| = \sqrt{2 + 4t^2}$$

$$\int_c z ds = \int_{-1}^0 (1+t)\sqrt{2+4t^2} dt = \int_{-1}^0 \sqrt{2+4t^2} dt + \int_{-1}^0 t\sqrt{2+4t^2} dt =$$

$$\left(u = 2t, a = \sqrt{2}, dt = \frac{1}{2} du \text{ OR } \sqrt{2+4t^2} = 2\sqrt{\frac{1}{2} + t^2}, a = \frac{1}{\sqrt{2}} \text{ Table} \right)$$

$$= \frac{1}{2} \left[t\sqrt{2+4t^2} + \ln(2t + \sqrt{2+4t^2}) \right]_{-1}^0 + \left[\frac{(2+4t^2)^{\frac{3}{2}}}{12} \right]_{-1}^0 =$$

$$= \frac{\sqrt{6}}{2} + \frac{1}{2} \ln \sqrt{2} - \frac{1}{2} \ln(\sqrt{6} - 2) + \left[\frac{\sqrt{2}}{6} - \frac{\sqrt{6}}{2} \right] = \sqrt{\frac{3}{2}} - \frac{1}{2} \ln(\sqrt{3} - \sqrt{2}).$$

5. Find $\int_c \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y, z) = (z, e^{\frac{y}{x}}, 2x)$ is given by $\mathbf{r}(t) = (t, t^2, e^t)$, $t \in [1, 2]$.

For 5)

$$\mathbf{r}(t) = (t, t^2, e^t), t \in [1, 2] \quad \mathbf{r}'(t) = (1, 2t, e^t)$$

then the field on $c : \mathbf{F} \circ \mathbf{r} = (e^t, e^t, 2t)$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_1^2 \mathbf{F} \cdot \mathbf{r}' dt = \int_1^2 (e^t + 4te^t) dt = (\text{by parts})$$

$$= [e^t + 4te^t - 4e^t]_1^2 = 5e^2 - e.$$

6. For $\mathbf{F}(x, y) = (3x\sqrt{x^2 + y^4} + \cos x, ky^3\sqrt{x^2 + y^4} + \sin y)$ find the value for k so that the field is conservative, then find a potential.

For 6)

$$F_1 = (3x\sqrt{x^2 + y^4} + \cos x) \quad F_2 = ky^3\sqrt{x^2 + y^4} + \sin y$$

for any point except the origin

$$(F_1)_y = \frac{6xy^3}{\sqrt{x^2 + y^4}} = (F_2)_x = \frac{kxy^3}{\sqrt{x^2 + y^4}} \text{ so } k = 6 \text{ then } f = ?$$

$$f_x = (3x\sqrt{x^2 + y^4} + \cos x) \text{ and } f_y = ky^3\sqrt{x^2 + y^4} + \sin y$$

$$f = \int f_x dx = \int (3x\sqrt{x^2 + y^4} + \cos x) dx + c(y) = (x^2 + y^4)^{\frac{3}{2}} + \sin x + c(y)$$

$$f_y = \frac{3}{2}\sqrt{x^2 + y^4} \cdot 4y^3 + c'(y) = F_2 \quad c'(y) = \sin y$$

$$\text{and } c(y) = -\cos y$$

$$\text{together } f(x, y) = (x^2 + y^4)^{\frac{3}{2}} + \sin x - \cos y + \text{const.}$$

7. Evaluate $\int_c z ds$ and c is given by $\mathbf{r}(t) = (t \cos t, t \sin t, t), t \in [0, 1]$.

For 7)

$$\mathbf{r}(t) = (t \cos t, t \sin t, t), t \in [0, 1] \quad \mathbf{r}'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$$

$$\|\mathbf{r}'(t)\| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} = \sqrt{2 + t^2}$$

$$\int_c z ds = \int_0^1 (t\sqrt{2 + t^2}) dt = \left[\frac{(2 + t^2)^{\frac{3}{2}}}{3} \right]_0^1 = \sqrt{3} - \frac{2\sqrt{2}}{3}.$$

8. Find $\int_c \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}(x, y, z) = (y, z, 2x - z)$ and c is the intersection of the plane $z = 2x$ and the paraboloid $z = x^2 + y^2$ oriented counterclockwise.

for 8)

$$\text{intersection of } z = 2x \text{ and } z = x^2 + y^2 \quad 2x = x^2 + y^2$$

$$1 = (x - 1)^2 + y^2 \quad x = 1 + \cos t, y = \sin t \text{ and } z = 2x$$

$$\mathbf{r}(t) = (1 + \cos t, \sin t, 2 + 2 \cos t), t \in [0, 2\pi]$$

$$\mathbf{r}'(t) = (-\sin t, \cos t, -2 \sin t)$$

$$\text{then the field on } c : \mathbf{F} \circ \mathbf{r} = (\sin t, 2 + 2 \cos t, 0)$$

$$\int_c \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{2\pi} (-\sin^2 t + 2 \cos t + 2 \cos^2 t) dt =$$

$$= -2\pi + [2 \sin t]_0^{2\pi} + \frac{3}{2} \int_0^{2\pi} (1 + \cos 2t) dt = -2\pi + 3\pi = \pi.$$