**The University of Calgary**
**Faculty of Science**
**Department of Mathematics and Statistics**
**MATH 353**
**Midterm Test**

**Time: 60 minutes** **Winter 2004**
**SOLUTION**

1. For $f(x,y) = 2x^3 - 3y^4 + 3x^2y$ find the absolute extrema inside and on the triangle with vertices $(0,0), (-2,2)$ and $(-2,0)$.

   for C.P. inside solve $\nabla f = \mathbf{0}$

   $6x^2 + 6xy = 6x(x+y) = 0$ so $x = 0$ or $y = -x$

   and $-12y^3 + 3x^2 = 0 \qquad x^2 = 4y^3$

   thus for $x = 0, y = 0$, for $x \neq 0$, $y = -x, x^2 = -4x^3, x = -\frac{1}{4}, y = \frac{1}{4}$

   two C.P. $\quad (0,0)$ and $\left(-\frac{1}{4}, \frac{1}{4}\right)$,actually on the boundary

   for C.P. on the boundary $\quad$ 3 parts

   $B_1 = \{x = -2, y \in [0,2]\}$ $f/_{B_1} = -16 - 3y^4 + 12y = g(y)$

   $g'(y) = -12(y^3 - 1) = 0$ for $y = 1$ and corners $(-2,0), (-2,1), (-2,2)$

   $B_2 = \{y = 0, x \in [-2,0]\}$ $f/_{B_2} = 2x^3 = h(x)$

   $h'(x) = 6x^2 = 0$ for $x = 0$ only corners

   $B_3 = \{y = -x, x \in [-2,0]\}$ $f/_{B_3} = -3x^4 - x^3 = h(x)$

   $h'(x) = -12x^3 - 3x^2 = -3x^2(4x+1) = 0$ for $x = 0$ , $-\frac{1}{4}$

   as above,nothing new

   all C.P. $\quad (0,0)$ ,$\left(-\frac{1}{4}, \frac{1}{4}\right), (-2,0), (-2,1), (-2,2)$

   values of $f \quad 0 \quad \frac{1}{256} \quad -7 \quad -16 \quad -40$

   thus

   maximum value is $\frac{1}{256}$ at $\left(-\frac{1}{4}, \frac{1}{4}\right)$, minimum value is $-40$ at $(-2,2)$.

2. Evaluate $\iint\limits_D \frac{x^2}{\sqrt{2x^2+y^2}}\, dxdy$ where $D = \{(x,y), 2x^2 + y^2 \leq 4, x \geq 0\}$.

   ( Hint: It is easier to use modified polar coord. than cartesian ones.)

   we want $2x^2 + y^2 = r^2$ so $x = \frac{1}{\sqrt{2}} r\cos\theta$ , $y = r\sin\theta, dxdy = \frac{1}{\sqrt{2}} rdrd\theta$

   and

   $$\iint\limits_D \frac{x^2}{\sqrt{2x^2+y^2}}\, dxdy = \iint\limits_{D^*} \frac{\frac{1}{2}r^2\cos^2\theta}{r} \frac{1}{\sqrt{2}} rdrd\theta$$

where $D^* = \{(r,\theta), 0 < r \le 2, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\}$

$$= \frac{1}{2\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \cdot \int_0^2 r^2 dr =$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta \cdot \left[\frac{1}{3}r^3\right]_0^2 = \frac{1}{2\sqrt{2}}\left[\frac{\pi}{2}+0\right]\frac{8}{3} = \frac{2\pi}{3\sqrt{2}.}$$

3. Describe/sketch the set $S$ in the coordinates $x, y, z$

   (a) if $S = \{\rho = 2r\}$
   where $\rho$ is from spherical , $r$ from cylindrical coordinates;

   (b) if $S = \{r = 1, \theta = \frac{\pi}{4}\}$ is given in cylindrical coordinates.

for a) $\rho^2 = x^2 + y^2 + z^2 = 4r^2 = 4x^2 + 4y^2$ means
$z^2 = 3(x^2 + y^2)$ two part cone

for b) $r = x^2 + y^2 = 1$ means a cylinder(shell) with radius 1, $z$ any

$\theta = \frac{\pi}{4}$ means half of a vertical plane $\{y = x, x > 0, z \text{ any}\}$

both means intersection so we get a vertical line

$\{x = y = \frac{1}{\sqrt{2}}, z \text{ any}\}$

4. Evaluate the integral $\iiint_B \frac{dxdydz}{\sqrt{x+2y+z}}$

where $B$ is the region in the first octant below the plane $x + 2y + z = 4$.

the solid $B = \{x \ge 0, y \ge 0, z \ge 0, x + 2y + z \le 4\}$

so bottom..$0 \le z \le 4 - x - 2y$ ...top

and necessarily $0 \le 4 - x - 2y$ so $D = \{x \ge 0, y \ge 0, x + 2y \le 4\}$

now,

$$\iiint_B \frac{dxdydz}{\sqrt{x+2y+z}} = \iint_D \left(\int_0^{4-x-2y} \frac{dz}{\sqrt{x+2y+z}}\right) dxdy = 2\iint_D \left[\sqrt{x+2y+z}\right]_{z=0}^{z=4-2y-x} dxdy$$

$$= 2\iint_D \left[2 - \sqrt{x+2y}\right] dxdy = 4 \text{ area of } D - 2\int_0^2 \left(\int_0^{4-2y} \sqrt{x+2y}dx\right) dy =$$

$$= 16 - 2\int_0^2 \left[\frac{2}{3}(x+2y)^{\frac{3}{2}}\right]_{x=0}^{x=4-2y} dy = 16 - \frac{4}{3}\int_0^2 \left[8 - (2y)^{\frac{3}{2}}\right] dy =$$

$$= 16 - \frac{64}{3} + \frac{4}{3}\left[\frac{1}{5}(2y)^{\frac{5}{2}}\right]_0^2 = \frac{-16}{3} + \frac{4 \cdot 2^5}{15} = \frac{16}{5}.$$

5. Set up the integral $I = \iiint\limits_B \frac{z}{\sqrt{x^2+y^2}} dxdydz$

where $B = \{(x, y, z)\,; x^2 + y^2 + z^2 \leq 2, z \geq x^2 + y^2, y \geq 0\}$

as iterated integrals in

(a) cylindrical coordinates;

(b) spherical soordinates        and evaluate only one from the above.

the solid is inside the sphere and above the parabolid

for a)

$$I = \iiint\limits_B \frac{z}{\sqrt{x^2+y^2}} dxdydz = \iiint\limits_{B^*} \frac{z}{r} r dr d\theta dz$$

where $B^* = \{r^2 + z^2 \leq 2, z \geq r^2, \theta \in [0, \pi]\}$

so $\quad I = \pi \iint\limits_{D^*} z dr dz$ where $D^* = \{0 < r \leq 1, r^2 < z < \sqrt{2 - r^2}\}$

since the intersection of the circle $r^2 + z^2 = 2$ and parabola $z = r^2$

is at $r = z = 1$

finally

$$I = \pi \int_0^1 \left( \int_{r^2}^{\sqrt{2-r^2}} z dz \right) dr = \frac{\pi}{2} \int_0^1 \left(2 - r^2 - r^4\right) dr = \frac{\pi}{2} \left(2 - \frac{1}{3} - \frac{1}{5}\right) = \frac{11\pi}{15}.$$

for b)

$$I = \iiint\limits_B \frac{z}{\sqrt{x^2+y^2}} dxdydz = \iiint\limits_{B^*} \frac{\rho \cos\phi}{\sqrt{\rho^2 \sin^2\phi}} \rho^2 \sin\phi d\rho d\theta d\phi = \iiint\limits_{B^*} \rho^2 \cos\phi d\rho d\theta d\phi$$

where $B^* = \{0 < \rho \leq \sqrt{2}, \rho\cos\phi \geq \rho^2 \sin^2\phi, \theta \in [0, \pi]\}$

so we got two upper limits for $\rho$,

the intersection of sphere and paraboloid: $z = 1, x^2 + y^2 = 1$

so $\phi = \frac{\pi}{4}$ and we have two parts

$$I = \pi \left[ \iint\limits_{D_1^*} \rho^2 \cos\phi d\rho d\theta d\phi + \iint\limits_{D_2^*} \rho^2 \cos\phi d\rho d\theta d\phi \right] \text{ where}$$

$D_1^* = \{\phi \in \left[0, \frac{\pi}{4}\right], 0 < \rho \leq \sqrt{2}\}$ and $D_2^* = \{\phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], 0 < \rho \leq \frac{\cos\phi}{\sin^2\phi}\}$.