THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 353 (L01)

Winter 2004 **SOLUTION** **TIME: 3 hours**

1. The temperature of a point on the sphere $x^2 + y^2 + z^2 = 50$
is given by $T(x, y, z) = 4xy + 3yz$.Find the coolest point(s) on the sphere.

For 1)

since T is a cont.function on a closed and bounded set we have to have
at least one max and one min points;for C.P. solve
$\nabla T = \lambda \nabla g$ where $g(x, y, z) = x^2 + y^2 + z^2 = 50$

$4y = \lambda 2x$

$4x + 3z = \lambda 2y$ in case of $\lambda \neq 0$ also $xyz \neq 0$ we get

$3y = \lambda 2z$

$$2\lambda = \frac{4y}{x} = \frac{4x + 3z}{y} = \frac{3y}{z} \text{ so from the first and last}$$

$4z = 3x$ and $16y^2 = 25x^2$ from the first and second

now back to the sphere $y = \pm\frac{5}{4}x, z = \frac{3}{4}x$ $x^2 = 16$

therefore 4 critical points $(\pm 4, \pm 5, \pm 3)$ and $(\pm 4, \mp 5, \pm 3)$

if $\lambda = 0$ then $y = 0$ and $4x + 3z = 0$ so $z = -\frac{4}{3}x$ back to the sphere

$\frac{25}{9}x^2 = 50$ $x = \pm 3\sqrt{2}$ and 2 more C.P.$\left(\pm 3\sqrt{2}, 0, \mp 4\sqrt{2}\right)$

compare values $T\left(\pm 3\sqrt{2}, 0, \mp 4\sqrt{2}\right) = 0, T(\pm 4, \pm 5, \pm 3) = 125$ max

and $T(\pm 4, \mp 5, \pm 3) = -125$ minimum and $(\pm 4, \mp 5, \pm 3)$ are the coolest pt.

2. Evaluate the iterated integral $\displaystyle\int_0^2 \left(\int_{y^2}^4 \frac{e^{\sqrt{x}}}{x}\, dx \right) dy$

by reversing the order of integration.

For 2)

given $0 \le y \le 2, y^2 \le x \le 4$ change to $0 \le x \le 4, 0 \le y \le \sqrt{x}$

then

$$\int_0^2 \left(\int_{y^2}^4 \frac{e^{\sqrt{x}}}{x}\, dx \right) dy = \int_0^4 \left(\frac{e^{\sqrt{x}}}{x} \int_0^{\sqrt{x}} dy \right) dx = \int_0^4 \left(\frac{e^{\sqrt{x}}}{\sqrt{x}} \right) dx = \left[2e^{\sqrt{x}}\right]_0^4 = 2\left(e^2 - 1\right).$$

3. Evaluate $$\iiint\limits_B \frac{xz\sqrt{x^2+y^2+z^2}dV}{\sqrt{x^2+y^2}}$$

where the solid B is the region in the first octant, outside the cylinder $x^2+y^2=1$, and inside the sphere $x^2+y^2+z^2=2$.

For 3)

$B=\{x^2+y^2+z^2\le 2, x^2+y^2\ge 1, x\ge 0, y\ge 0, z\ge 0\}$

by cylindr.coord.

$$\iiint\limits_B \frac{xz\sqrt{x^2+y^2+z^2}dxdydz}{\sqrt{x^2+y^2}} = \iiint\limits_{B^*} \frac{zr\cos\theta\sqrt{r^2+z^2}rdrdzd\theta}{r} \text{ where}$$

$B^*=\{r^2+z^2\le 2, r\ge 1, \theta\in\left[0,\frac{\pi}{2}\right], z\ge 0\}$

it means $1\le r\le\sqrt{2}$ and $0\le z\le\sqrt{2-r^2}$

OR $0\le z\le 1$ and $1\le r\le\sqrt{2-z^2}$

so the integral $$I=\int\limits_0^{\frac{\pi}{2}}\cos\theta d\theta\int\limits_1^{\sqrt{2}} r\left(\int\limits_0^{\sqrt{2-r^2}} z\sqrt{r^2+z^2}dz\right)dr = [\sin\theta]_0^{\frac{\pi}{2}}\int\limits_1^{\sqrt{2}} r\left[\tfrac{1}{3}\left(r^2+z^2\right)^{\frac{3}{2}}\right]_{z=0}^{z=\sqrt{2-r^2}} dr =$$

$$=\tfrac{1}{3}\int\limits_1^{\sqrt{2}} r\left[2\sqrt{2}-r^3\right]dr = \frac{2\sqrt{2}}{3}\left[\tfrac{r^2}{2}\right]_1^{\sqrt{2}} - \left[\tfrac{r^5}{15}\right]_1^{\sqrt{2}} = \tfrac{\sqrt{2}}{3}-\tfrac{4\sqrt{2}}{15}+\tfrac{1}{15}=\frac{1+\sqrt{2}}{15}$$

OR spherical coord.

$$\iiint\limits_B \frac{xz\sqrt{x^2+y^2+z^2}dxdydz}{\sqrt{x^2+y^2}} = \iiint\limits_{B^*} \frac{\rho\cos\theta\sin\phi\rho\cos\phi\rho^3\sin\phi d\rho d\phi d\theta}{\rho\sin\phi} = \iiint\limits_{B^*} \cos\theta\rho^4\cos\phi\sin\phi d\rho d$$

where

$0\le\rho\le\sqrt{2}, \rho\sin\phi\ge 1, \theta\in\left[0,\frac{\pi}{2}\right], \phi\in\left[0,\frac{\pi}{2}\right]$ so $\frac{1}{\sin\phi}\le\rho\le\sqrt{2}$ and

necessarily $\sin\phi\ge\frac{1}{\sqrt{2}}$ $\phi\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

$B^*=\{\frac{1}{\sin\phi}\le\rho\le\sqrt{2}, \phi\in\left[\frac{\pi}{4},\frac{\pi}{2}\right], \theta\in\left[0,\frac{\pi}{2}\right]\}$

$$\text{so the integral} = \int\limits_0^{\frac{\pi}{2}}\cos\theta d\theta\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin\phi\cos\phi\left(\int\limits_{\frac{1}{\sin\phi}}^{\sqrt{2}}\rho^4 d\rho\right)d\phi = \frac{1}{5}\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\phi\sin\phi\left(4\sqrt{2}-\frac{1}{\sin^5\phi}\right)d\phi =$$

$$=\frac{\sqrt{2}}{5}\left[-\cos 2\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \tfrac{1}{15}\left[\sin^{-3}\phi\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \ldots$$

4. Find $\displaystyle\iint_S z\, dS$ where S is the part of $z = \sqrt{x+y}$ in the first octant between planes $z = 2$ and $z = 0$.

For 4)

$0 \le z \le 2$ gives $0 \le x + y \le 4$ so $z = \sqrt{x+y}$ for $(x, y) \in D$

where $D = \{0 \le x + y \le 4, x \ge 0, y \ge 0\}$ the region is a triangle

then

$$\mathbf{n} = \left(\frac{1}{2\sqrt{x+y}}, \frac{1}{2\sqrt{x+y}}, -1\right) \text{ and } \|\mathbf{n}\| = \sqrt{\frac{1+2x+2y}{2(x+y)}} = \frac{\sqrt{1+2x+2y}}{\sqrt{2}\sqrt{x+y}}$$

$$\text{so } \iint_S z\, dS = \iint_D \sqrt{x+y}\ \|\mathbf{n}\|\, dxdy = \frac{1}{\sqrt{2}}\iint_D \sqrt{1+2x+2y}dxdy =$$

$$= \frac{1}{\sqrt{2}}\iint_{T_1} \sqrt{1+2x+2y}dxdy$$

where $D = \{0 \le x \le 4, 0 \le y \le 4 - x\}$

$$\tfrac{1}{\sqrt{2}}\iint_D \sqrt{1+2x+2y}dxdy = \tfrac{1}{\sqrt{2}}\int_0^4 \left(\int_0^{4-x} \sqrt{1+2x+2y}dy\right) dx = \frac{1}{3\sqrt{2}}\int_0^4 \left[(1+2x+2y)^{\frac{3}{2}}\right]_{y=0}^{y=4-x} dx =$$

$$= \frac{1}{3\sqrt{2}}\int_0^4 \left[9^{\frac{3}{2}} - (1+2x)^{\frac{3}{2}}\right] dx = 18\sqrt{2} - \frac{1}{15\sqrt{2}}\left[(1+2x)^{\frac{5}{2}}\right]_0^4 = 18\sqrt{2} - \frac{121\sqrt{2}}{15} =$$

$$\frac{149\sqrt{2}}{15}$$

5. Find $\displaystyle\oint_c \mathbf{F} \bullet \mathbf{ds}$

where $\mathbf{F}(x, y) = (5y^2 + e^{x^3}, 7x^2 + \cos(\pi y))$ and c is the closed curve consisting from 3 line segments

from $(0, 0)$ to $(2, 0)$ to $(2, 4)$ and back to $(0, 0)$.

For 5)

the curve is closed so by Green's Th.

$$\oint_c \mathbf{F} \bullet \mathbf{ds} = \iint_D \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right] dxdy = \iint_D [14x - 10y]\, dxdy$$

where $D = \{0 \le x \le 2, 0 \le y \le 2x\} = \{0 \le y \le 4, \frac{y}{2} \le x \le 2\}$

so

$$\oint_c \mathbf{F} \bullet \mathbf{ds} = 14\int_0^2 x\left(\int_0^{2x} dy\right)dx - 10\int_0^4 y\left(\int_{\frac{y}{2}}^2 dx\right)dy = 28\int_0^2 x^2dx - 10\int_0^4 \left(2y - \tfrac{y^2}{2}\right) dy =$$

$= 28 \cdot \frac{8}{3} - 160 + 40 \cdot \frac{8}{3} = \dfrac{64}{3}$

6. For the vector field $\mathbf{F}(x, y, z) = \left(\arcsin\left(\frac{y}{z}\right), \ln(1 + x^2 + y^2), ye^{xz^2}\right)$
   calculate $div\mathbf{F}$ and $curl\mathbf{F}$ in the domain.

   For 6)

   for $z \neq 0$ and $|y| < |z|$ all functions defined and differentiable

   so $$div\mathbf{F} = \frac{\partial}{\partial x}\left(\arcsin\frac{y}{z}\right) + \frac{\partial}{\partial y}\left(\ln(1 + x^2 + y^2)\right) + \frac{\partial}{\partial z}\left(ye^{xz^2}\right) =$$
   $$= 0 + \frac{2y}{1 + y^2 + z^2} + 2xyze^{xz^2}$$

   since $(\arcsin u)' = \dfrac{u'}{\sqrt{1-u^2}}$ $$curl\mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arcsin\frac{y}{z} & \ln(1 + y^2 + x^2) & ye^{xz^2} \end{bmatrix} =$$

   $$\left(e^{xz^2} - 0, -yz^2e^{xz^2} + \frac{-y}{z^2\sqrt{1 - \left(\frac{y}{z}\right)^2}}, \frac{2x}{1 + x^2 + y^2} - \frac{1}{z\sqrt{1 - \left(\frac{y}{z}\right)^2}}\right) =$$
   $$= \left(e^{xz^2}, \frac{-y}{|z|\sqrt{z^2 - y^2}} - yz^2e^{xz^2}, \frac{2x}{1 + x^2 + y^2} - \frac{sgn z}{\sqrt{z^2 - y^2}}\right).$$

7. Evaluate $\displaystyle\int_{\mathbf{c}} \mathbf{F} \bullet \mathbf{ds}$ where $\mathbf{F}(x, y, z) = (yz\cos(xy), xz\cos(xy), \sin(xy))$

   and $\mathbf{c}$ is the line segment from $A\left(\pi, \frac{1}{2}, 1\right)$ to $B\left(\frac{-1}{2}, \pi, 2\right)$.

   For 7)

   find a potential f such that $\mathbf{F} = \nabla f$

   $f_x = yz\cos(xy)$ $f_y = xz\cos(xy)$ $f_z = \sin(xy)$

   ....

   we can see that $f(x, y, z) = z\sin(xy)$

   and $\displaystyle\int_{\mathbf{c}} \mathbf{F} \bullet \mathbf{ds} = f(B) - f(A) = 2\sin\frac{-\pi}{2} - \sin\frac{\pi}{2} = -3.$

8. Find the flux of $\mathbf{F} = (x^2 - 3x, y^2 + 2y, z^2)$ outward
   from the closed surface $S = \{x^2 + y^2 + z^2 = 9\}$
   Use the Divergence Theorem.

   For 8)

$div\mathbf{F} = 2x - 3 + 2y + 2 + 2z$ and $\iint\limits_S \mathbf{F} \bullet \mathbf{dS} = \iiint\limits_B (2x + 2y - 1 + 2z)\, dxdydz$

where $\qquad B = \{x^2 + y^2 + z^2 \leq 9\}$ THANKS TO SYMMETRY

$\iint\limits_S \mathbf{F} \bullet \mathbf{dS} =$ (odd f. in x.,y,z) 0−volume of $B = -\frac{4}{3}\pi R^3 = -36\pi$

it means that flux is flowing in

9. Evaluate $\oint\limits_c \mathbf{F} \bullet \mathbf{ds}$ where $\mathbf{F}(x, y, z) = (4xy + ze^x, y, yz)$ and

   the closed counterclockwise oriented curve c is the intersection

   of the plane $z = 2x + 3$ and the paraboloid $\quad z = x^2 + 2y^2$.

   Use Stokes' Theorem.

   For 9)

   we can define S as a part of the plane inside paraboloid so

   $z = 2x + 3$ for $(x, y) \in D$ where $2x + 3 \geq x^2 + 2y^2$

   gives

   $D = \{(x-1)^2 + 2y^2 \leq 4\} = \{\left(\frac{x-1}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 \leq 1\}$

   then

   $\mathbf{n} = (-\nabla z, 1) = (-2, 0, 1)$ upward

   we need $curl\mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ 4xy + ze^x & y & yz \end{bmatrix} = (z, e^x, -4x) = (2x + 3, e^x, -4x)$ on S

   $\oint\limits_c \mathbf{F} \bullet \mathbf{ds} = \iint\limits_S curl\mathbf{F} \bullet \mathbf{dS} = \iint\limits_D curl\mathbf{F} \bullet \mathbf{n}\; dxdy = \iint\limits_D (-8x - 6)\, dxdy =$

   $= -8\iint\limits_D (x-1) dxdy - 14$area of ellipse$=0 - 14\pi 2\sqrt{2} = -28\sqrt{2}\pi$

   Or

   modifies poloar coord. $\quad x = 1 + 2r\cos\theta \quad y = \sqrt{2}r\sin\theta \quad dxdy = 2\sqrt{2}rdrd\theta$

   $0 \leq r \leq 1 \quad \theta \in [0, 2\pi]$ so

   $= -16\sqrt{2}\iint\limits_{D^*} (r + 2r^2\cos\theta)\, drd\theta - 6\pi ab = -28\pi\sqrt{2}.$

10. Prove that for any smooth vector field $\mathbf{F}(x, y, z)$

    $div(curl\mathbf{F}) = 0 \qquad$ OR $\qquad \boldsymbol{\nabla} \bullet (\boldsymbol{\nabla} \times \mathbf{F}) = 0.$

For 10)

$$\mathbf{G} = curl\mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{bmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

then

$$div(curl\mathbf{F}) = div(\mathbf{G}) = (G_1)_x + (G_2)_y + (G_3)\,z =$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right)_x + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right)_y + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)_z =$$

$$= (F_3)_{yx} - (F_2)_{zx} + (F_1)_{zy} - (F_3)_{xy} + (F_2)_{xz} - (F_1)_{zy} = 0$$

since all functions are continuous ,thus diff.does not depend on the order.