MATH 353 FINAL HANDOUT-SOLUTION

For 1) $f(x,y) = x^2(y+1)^3 + y^2$

$f_x = 2x(y+1)^3 = 0 \qquad f_y = 3x^2(y+1)^2 + 2y = 0$

if $x = 0$ then $y = 0$;if $y = -1$ then No sol.

the only C.P. is $(0,0)$,now,Second.der.test

$f_{xx} = 2(y+1)^3 \qquad A = 2$

$f_{xy} = 6x(y+1)^2 \qquad B = 0$

$f_{yy} = 6x^2(y+1) + 2 \qquad C = 2 \qquad D = -4$

$D < 0, A > 0$ loc. minimum

For abs. min. we have to show that $f(x,y) \geq f(0,0) = 0$ for all (x,y)

BUT $f(R,-2) = -R^2 + 4 < 0$ for big R

thus it is NOT abs. min.

For 2)

$\int_{-1}^{0}(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4+1})dy$.given $\qquad -1 \leq y \leq 0 \qquad -1 \leq x \leq \sqrt[3]{y}$

change the order $\qquad -1 \leq x \leq 0 \qquad x^3 \leq y \leq 0$

then

$$\int_{-1}^{0}(\int_{-1}^{\sqrt[3]{y}} \frac{dx}{x^4+1})dy = \qquad \int_{-1}^{0}(\frac{1}{x^4+1}\int_{x^3}^{0} dy)dx =$$

$$= \int_{-1}^{0} \frac{-x^3dx}{x^4+1} = \frac{1}{4}\left[-\ln(x^4+1)\right]_{-1}^{0} = \frac{\ln 2}{4}.$$

For 3)

on the sphere $x^2 + y^2 + z^2 = 36$ closest to $P(1,2,2)$.

distance is $\sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$ so looking for minimum of

$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2$ with $g(x,y,z) = x^2 + y^2 + z^2 = 36$

$\nabla f = \lambda \nabla g \qquad 2(x-1) = \lambda 2x \qquad 2(y-2) = \lambda 2y \qquad 2(z-2) = \lambda 2z$

gives $\lambda = \frac{x-1}{x} = \frac{y-2}{y} = \frac{z-2}{z}$ since $\lambda xyz \neq 0$

thus $z = y$ and $xy - y = yx - 2x \qquad y = 2x$ back to the sphere

$x^2 + 4x^2 + 4x^2 = 9x^2 = 36 \qquad x = \pm 2$ and we have 2 C.P.

for $(-2,-4,-4)$ distance is $\sqrt{9+36+36} = 9$ and

$(2,4,4)$ is t**he closest poin**t since distance is 3 so minimum

For 4 a)

$D = \{\ x + y \geq 2$ and $x^2 + y^2 \leq 4\} = \{0 \leq x \leq 2, 2 - x \leq y \leq \sqrt{4-x^2}\}$

so

$$\iint_D \frac{x+y}{x^2+y^2}dxdy = \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{x+y}{x^2+y^2}dy\right)dx = \text{OR} = \int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} \frac{x+y}{x^2+y^2}dx\right)dy$$

for b)

$D^* = \{r(\cos\theta + \sin\theta) \geq 2$ and $0 \leq r \leq 2, \theta \in \left[0, \frac{\pi}{2}\right]\}$

$$\iint\limits_{D}\frac{x+y}{x^2+y^2}dxdy=\iint\limits_{D^*}\frac{r(\cos\theta+\sin\theta)}{r^2}rdrd\theta=\int\limits_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\left(\int\limits_{\frac{2}{\cos\theta+\sin\theta}}^{2}dr\right)d\theta=$$

$$=\int\limits_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\left(2-\frac{2}{\cos\theta+\sin\theta}\right)d\theta=\int\limits_0^{\frac{\pi}{2}}2(\cos\theta+\sin\theta)\,d\theta-2\cdot\frac{\pi}{2}=$$

$$=2\left[\sin\theta-\cos\theta\right]_0^{\frac{\pi}{2}}-\pi=4-\pi.$$

For 5) the set B is a layer of a full paraboloid
use cylindrical coordinates
$B^*=\{1\le z\le 4 \text{ and } z\ge r^2,\theta\in[0,2\pi]\}$

so $$\iiint\limits_{B}\frac{zdV}{\sqrt{x^2+y^2}}=\iiint\limits_{B^*}\frac{z}{\sqrt{r^2}}rdrdzd\theta=2\pi\iint\limits_{D^*}zdrdz$$

where $D^*=\{1\le z\le 4\ ,0\le r\le\sqrt{z}\}$

the integral $$I=2\pi\int\limits_1^4 z(\int\limits_0^{\sqrt{z}}dr)dz=2\pi\int\limits_1^4 z^{\frac{3}{2}}dz=\frac{4\pi}{5}\left[z^{\frac{5}{2}}\right]_1^4=\frac{4\pi}{5}\left[2^5-1\right]=\frac{124\pi}{5}.$$

For 6) $S=\{\ z=\sqrt{3x^2+3y^2}\quad x+z\le 4\}$

$$SA=\iint\limits_{S}dS=\iint\limits_{D}\|\mathbf{n}\|\,dxdy$$

$$\mathbf{n}=(\nabla z,-1)=\left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}},\frac{\sqrt{3}y}{\sqrt{x^2+y^2}},-1\right)\text{or}\left(\frac{3x}{\sqrt{3x^2+3y^2}},\frac{3y}{\sqrt{3x^2+3y^2}},-1\right)$$

and $\|\mathbf{n}\|^2=\dfrac{3x^2+3y^2}{x^2+y^2}+1=4\qquad\|\mathbf{n}\|=2$

we have to find D the intersection of the given cone and plane
$z=4-x=\sqrt{3x^2+3y^2}\qquad 16-8x+x^2=3x^2+3y^2$
$16=2x^2+8x+3y^2\qquad 24=2(x+2)^2+3y^2$

finally $$D=\{2(x+2)^2+3y^2\le 24\}=\{\left(\frac{x+2}{\sqrt{12}}\right)^2+\left(\frac{y}{\sqrt{8}}\right)^2\le 1\}$$

so $SA=2\cdot area$ of an ellipse $=2\pi ab=2\pi\sqrt{12}\sqrt{8}=8\pi\sqrt{6}.$

For 7)
the curve is closed,oriented clockwise, $n=2$ use Green's theorem

$\oint\limits_c\mathbf{F}\cdot\mathbf{ds}=-\iint\limits_{T}\left[(F_2)_x-(F_1)_y\right]dxdy$ because of the orientation of c

where $T=\{0\le x\le 2,2-x\le y\le 2\}=\{0\le y\le 2,2-y\le x\le 2\}$

and $\left[(F_2)_x-(F_1)_y\right]=\pi\cos\pi x-3xy^2$

$$=-\iint\limits_{T}\left[\pi\cos\pi x-3xy^2\right]dxdy=\int\limits_0^2 3y^2(\int\limits_{2-y}^{2}xdx)dy-\int\limits_0^2\pi\cos\pi x(\int\limits_{2-x}^{2}dy)dx=$$

$$=\int\limits_0^2 3y^2\cdot\frac{2^2-(2-y)^2}{2}dy-\int\limits_0^2 x\cdot\pi\cos\pi xdx=\frac{3}{2}\int\limits_0^2\left(4y^3-y^4\right)dy-\left[x\sin\pi x\right]_0^2-\left[\frac{\cos\pi x}{\pi}\right]_0^2=$$

$$=\frac{3}{2}\left[2^4-\frac{2^5}{5}\right]-0=\frac{72}{5}.$$

For 8)

$div(\phi\mathbf{F}) = (\phi F_1)_x + (\phi F_2)_y = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 =$
$= \phi(F_1)_x + \phi(F_2)_y + (\phi_x, \phi_y) \bullet (F_1, F_2) = \phi div\mathbf{F} + grad\phi \bullet \mathbf{F}$

.**For 9)**

since c is closed ,$n = 3$.we can use Stokes's Theorem

$$I = \oint_c \mathbf{F} \cdot \mathbf{ds} = \iint_S curl\ \mathbf{F} \cdot \mathbf{dS}$$

where

$$curl\ \mathbf{F} = \left\| \begin{matrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2 + y & y^3 - x & z^4 \end{matrix} \right\| = (0, 0, -2)$$

and

S is a part of the plane $z = 2 - 2x + 3y$ so $\mathbf{n} = (-\nabla z, 1) = (2, -3, 1)$
inside the cylinder $\{x^2 + y^2 \le 4\} = D$ $\quad curl\ \mathbf{F} \cdot \mathbf{n} = -2$

thus $\quad I = \iint_D (-2)\, dxdy = -2 \cdot$area of $D = -8\pi.$

For 10) since S is closed use Divergence theorem

$div\mathbf{F} = 2x + 2y + 2z$ and

$$I = \iint_S \mathbf{F} \cdot \mathbf{dS} = 2\iiint_B (x + y + z)\, dxdydz \text{ where } B = \{x^2 + y^2 + 4(z-1)^2 \le 4\}$$

ellipsoid symmetrical in x and in y so $I = 0 + 0 + 2\iiint_B zdxdydz =$

$$= 2\int_0^2 z \iint_{D_z} dxdy$$

where $D_z = \{x^2 + y^2 \le 4 - 4(z-1)^2 = 8z - 4z^2\}$ circular disc for each fixed z
with area $= \pi\,(8z - 4z^2)$

thus $$I = 2\int_0^2 z\pi\,[8z - 4z^2]\, dz = 8\pi \int_0^2 (2z^2 - z^3) dz = 8\pi \left[\frac{2}{3} \cdot 2^3 - \frac{2^4}{4}\right] = \frac{32}{3}\pi$$

OR cylindrical coordinates $2\iiint_B zdxdydz = 4\pi \iint_{D^*} zrdrdz$

where $D^* = \{r^2 + 4\,(z-1)^2 \le 4\}$

$$\text{the integral} = 4\pi \int_0^2 z \left(\int_0^{\sqrt{4-4(z-1)^2}} rdr \right) dz = 8\pi \int_0^2 z\,(2z - z^2)\, dz = ...$$

Or modified spherical coord.$z = 1 + 2\rho\cos\phi$..

For 11) the curve is not closed but the field is conservative
so we can find a potential

:$f_x = y \qquad f_y = x \qquad f_z = 2z$

$f\,(x, y, z) = xy + z^2$ and $\int_c \mathbf{F} \bullet \mathbf{ds} = f(B) - f(A) = 5 - 3 = 2$

OR find a parametrization of c

from the cylinder $\quad x = \sqrt{2}\cos t,\ y = \sqrt{2}\sin t$ and from $z = 2xy$
we have $\quad \mathbf{r}\,(t) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 4\cos t \sin t\right) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 2\sin 2t\right)$

now for $A \qquad t=\frac{3}{4}\pi$, and for $B \qquad t=\frac{\pi}{4}$

$\mathbf{r}'(t)=\left(-\sqrt{2}\sin t,\sqrt{2}\cos t,4\cos 2t\right)$ and the field on c

$\mathbf{F}=(y,x,2z)_c=\left(\sqrt{2}\sin t,\sqrt{2}\cos t,4\sin 2t\right)$ then

$\mathbf{F}\cdot\mathbf{r}'=-2\sin^2 t+2\cos^2 t+8\sin 4t$

$$\int_c \mathbf{F}\cdot\mathbf{ds}=\int_{\frac{3}{4}\pi}^{\frac{\pi}{4}}\mathbf{F}\cdot\mathbf{r}'dt=-\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}\left[2\cos 2t+8\sin 4t\right]dt=$$

$$=\left[-\sin 2t+2\cos 4t\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}=1+1-2+2=2.$$

For 12)

$\mathbf{F}(x,y,z)=\left(\arctan z(x^2+y^2),\ln(1+y^2+z^2),y\ e^{xyz}\right)$

for any (x,y,z)

$$div\mathbf{F}=\frac{2xz}{1+z^2\left(x^2+y^2\right)^2}+\frac{2y}{1+y^2+z^2}+xy^2\ e^{xyz}$$

and $\qquad curl\mathbf{F}=\begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arctan z(x^2+y^2) & \ln(1+y^2+z^2) & ye^{xyz}\end{bmatrix}=$

$$=\left(e^{xyz}\left(1+xyz\right)-\frac{2z}{1+y^2+z^2},-y^2ze^{xyz}+\frac{x^2+y^2}{1+z^2\left(x^2+y^2\right)^2},-\frac{2yz}{1+z^2\left(x^2+y^2\right)^2}\right).$$