FINAL EXAMINATION MATH 353 -02

WINTER 2002 TIME: 3 hours SOLUTION

1. For $f(x,y) = x^2y - 6y$ find
   (a) local extrema and
   (b) absolute extrema on $D = \{(x,y)\,;x^2+y^2 \le 9\}$. [15]

for a)
solve $\nabla f = \mathbf{0}$ $f_x = 2xy = 0$ and $f_y = x^2 - 6 = 0$
only two points $\left(\pm\sqrt{6}, 0\right)$
now by Second Derivative test
$f_{xx} = 2y = 0$ $f_{xy} = 2x$ $f_{yy} = 0$
the discriminant $D = f_{xy}^2 - f_{xx}f_{yy} = 4x^2 > 0$ both are **saddle** points
for b)
inside critical points as in a)
on the boundary $\partial D = \{(x,y)\,;g(x,y) = x^2+y^2 = 9\}$
$x^2 = 9 - y^2$ f on $\partial D = h(y) = 3y - y^3$ on$[-3,3]$
end points $(0,\pm 3)$ and critical points $\left(\pm\sqrt{8},1\right)$, $\left(\pm\sqrt{8},-1\right)$
since $h'(y) = 3 - 3y^2 = 0$ for $y = \pm 1$
OR
use Langrange multiplier $\nabla f = \lambda \nabla g$
solve $2xy = \lambda 2x$ $x^2 - 6 = \lambda 2y$ $x^2+y^2 = 9$
from the first equ. $x = 0$ back to the constraint $y = \pm 3$ $(\lambda = \mp 1)$
OR for $x \ne 0$ $y = \lambda$ $x^2 - 6 = 2y^2$
back to the constraint $6 + 3y^2 = 9$ so $y = \pm 1, x = \pm\sqrt{8}$
we have 6 critical points on the boundary,now compare values
$f(\pm\sqrt{8},1) = 2$ $f(\pm\sqrt{8},-1) = -2$ and
$f(0,3) = -18$ min $f(0,-3) = 18$ max.

2. Express the integral

$$\iint\limits_D \frac{x+y}{x^2+y^2}dxdy$$

where D is the region above the line $x+y=2$ and inside the circle $x^2+y^2=4$

   (a) as interated integrals in cartesian coordinates;

(b) as iterated integrals in polar coordinates

then evaluate only once. [10]

for a)

sketch the set

$D = \{\ x+y \geq 2 \text{ and } x^2+y^2 \leq 4\} = \{0 \leq x \leq 2, 2-x \leq y \leq \sqrt{4-x^2}\}$

so

$$\iint_D \frac{x+y}{x^2+y^2} dxdy = \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{x+y}{x^2+y^2} dy \right) dx = \text{OR} = \int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} \frac{x+y}{x^2+y^2} dx \right) dy$$

for b)

$D^* = \{r(\cos\theta + \sin\theta) \geq 2 \text{ and } 0 \leq r \leq 2, \theta \in \left[0, \frac{\pi}{2}\right]\}$

$$\iint_D \frac{x+y}{x^2+y^2} dxdy = \iint_{D^*} \frac{r(\cos\theta + \sin\theta)}{r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(\int_{\frac{2}{\cos\theta + \sin\theta}}^{2} dr \right) d\theta =$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta) \left(2 - \frac{2}{\cos\theta + \sin\theta} \right) d\theta = \int_0^{\frac{\pi}{2}} 2 (\cos\theta + \sin\theta)\, d\theta - 2 \cdot \frac{\pi}{2} =$$

$$= 2 \left[\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}} - \pi = 4 - \pi.$$

3. Evaluate $\iiint_B \frac{zdV}{\sqrt{x^2+y^2}}$

where $B = \{(x, y, z)\,; 1 \leq z \leq 4 \text{ and } z \geq x^2 + y^2\}$.

sketch the set B a layer of a paraboloid

use cylindrical coordinates

$B^* = \{1 \leq z \leq 4 \text{ and } z \geq r^2, \theta \in [0, 2\pi]\}$

so $$\iiint_B \frac{zdV}{\sqrt{x^2+y^2}} = \iiint_{B^*} \frac{z}{\sqrt{r^2}} r dr dz d\theta = 2\pi \iint_{D^*} z dr dz$$

where $D^* = \{1 \leq z \leq 4\ , 0 \leq r \leq \sqrt{z}\}$

the integral $I = 2\pi \int_1^4 z(\int_0^{\sqrt{z}} dr) dz = 2\pi \int_1^4 z^{\frac{3}{2}} dz = \frac{4\pi}{5} \left[z^{\frac{5}{2}}\right]_1^4 = \frac{4\pi}{5} [2^5 - 1] = \frac{124\pi}{5}.$

4. Find the surface area of S

where S is the part of $z = \sqrt{3x^2 + 3y^2}$ below the plane $x + z = 4$.

$SA = \iint\limits_S dS = \iint\limits_D \|\mathbf{n}\|\, dxdy$ we have to find D and $\mathbf{n} = (\nabla z, -1) = \left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}, \frac{\sqrt{3}y}{\sqrt{x^2+y^2}}, -1 \right)$

and $\|\mathbf{n}\|^2 = \frac{3x^2+3y^2}{x^2+y^2} + 1 = 4 \qquad \|\mathbf{n}\| = 2$

for D find the intersection of the given cone and plane

$z = 4 - x = \sqrt{3x^2 + 3y^2} \qquad 16 - 8x + x^2 = 3x^2 + 3y^2$

$16 = 2x^2 + 8x + 3y^2 \qquad 24 = 2(x+2)^2 + 3y^2$

finally $\quad D = \{2(x+2)^2 + 3y^2 \le 24\} = \{\left(\frac{x+2}{\sqrt{12}}\right)^2 + \left(\frac{y}{\sqrt{8}}\right)^2 \le 1\}$

so $\quad SA = 2 \cdot area$ of an ellipse $= 2\pi ab = 2\pi\sqrt{12}\sqrt{8} = 8\pi\sqrt{6}$.

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5. Find

$$\oint\limits_c \mathbf{F} \cdot \mathbf{ds}$$

where $\mathbf{F} = (y^3x + \cos(x^2), e^{y^2} + \sin(\pi x))$ and c is boundary of the triangle T from $(0,2)$ to $(2,2)$ to $(2,0)$ and back to $(0,2)$. [10]

the curve is closed and $n = 2$ use Green's theorem

$\oint\limits_c \mathbf{F} \cdot \mathbf{ds} = -\iint\limits_T \left[(F_2)_x - (F_1)_y\right] dxdy$ because of the orientation of c

where $T = \{0 \le x \le 2, 2 - x \le y \le 2\} = \{0 \le y \le 2, 2 - y \le x \le 2\}$

and $\quad \left[(F_2)_x - (F_1)_y\right] = \pi \cos \pi x - 3xy^2$

so the integral $\quad = -\iint\limits_T [\pi \cos \pi x - 3xy^2]\, dxdy = \int_0^2 3y^2 (\int_{2-y}^{2} xdx)dy - \int_0^2 \pi \cos \pi x (\int_{2-x}^{2} dy)dx =$

$= \int_0^2 3y^2 \cdot \frac{2^2 - (2-y)^2}{2} dy - \int_0^2 x \cdot \pi \cos \pi x dx = \frac{3}{2} \int_0^2 (4y^3 - y^4)\, dy - [x \sin \pi x]_0^2 - \left[\frac{\cos \pi x}{\pi}\right]_0^2 =$

$= \frac{3}{2}\left[2^4 - \frac{2^5}{5}\right] - 0 = \frac{72}{5}.$

6. Show that for any smooth vector field $\mathbf{F}(x, y)$ and any smooth real-velude function $\phi(x, y)$

$div\,(\phi\mathbf{F}) = grad\phi \cdot \mathbf{F} + \phi div\ \mathbf{F} \qquad \boldsymbol{\nabla} \cdot (\phi\mathbf{F}) = \boldsymbol{\nabla\phi} \cdot \mathbf{F} + \phi\,(\boldsymbol{\nabla} \cdot \mathbf{F})$. [10]

$div(\phi\mathbf{F}) = (\phi F_1)_x + (\phi F_2)_y + (\phi F_3)_z = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 + \phi(F_3)_z + \phi_z F_3 =$
$= \phi(F_1)_x + \phi(F_2)_y + \phi(F_3)_z + (\phi_x, \phi_y, \phi_z) \cdot F_1, F_2, F_3) = \phi div\mathbf{F} + grad\phi \cdot \mathbf{F}$

7. Evaluate $\displaystyle\oint_c \mathbf{F} \cdot \mathbf{ds}$ where $\mathbf{F} = (x^2 + y, y^3 - x, z^4)$

and c is given as $\{x^2 + y^2 = 4\} \cap \{2x - 3y + z = 2\}$ oriented positively. [10]

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since c is closed we can use Stokes's Theorem

$$I = \oint_c \mathbf{F} \cdot \mathbf{ds} = \iint_S curl\ \mathbf{F} \cdot \mathbf{dS}$$

where

$$curl\ \mathbf{F} = \begin{Vmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2 + y & y^3 - x & z^4 \end{Vmatrix} = (0, 0, -2)$$

and

S is a part of the plane $z = 2 - 2x + 3y$ so $\mathbf{n} = (-\nabla z, 1) = (2, -3, 1)$
inside the cylinder $\{x^2 + y^2 \leq 4\} = D$
thus

$$I = \iint_D (-2)\, dxdy = -2 \cdot \text{area of } D = -8\pi.$$

8. Find the flux of $\mathbf{F} = (x^2, y^2, z^2)$ outward
from the closed surface $S = \{x^2 + y^2 + 4(z-1)^2 = 4\}$. [10]

since S is closed use Divergence theorem
$div\mathbf{F} = 2x + 2y + 2z$ and

$$I = \iint_S \mathbf{F} \cdot \mathbf{dS} = 2\iiint_B (x + y + z)\, dxdydz \text{ where } B = \{x^2 + y^2 + 4(z-1)^2 \leq 4\}$$

ellipsoid symmetrical in x and in y so $I = 0 + 0 + 2\displaystyle\iiint_B zdxdydz =$

$$= 2\int_0^2 z \iint_{D_z} dxdy$$

where $D_z = \{x^2 + y^2 \leq 4 - 4(z-1)^2 = 8z - 4z^2\}$ circle for each fixed z

thus $$I = 2\int_0^2 z\pi \left[8z - 4z^2\right] dz = 8\pi \int_0^2 (2z^2 - z^3) dz = 8\pi \left[\frac{2}{3} \cdot 2^3 - \frac{2^4}{4}\right] = \frac{32}{3}\pi$$

Or cylindrical coordinates $2\iiint\limits_B zdxdydz = 4\pi \iint\limits_{D^*} zrdrdz$

where $D^* = \{r^2 + 4(z-1)^2 \le 4\}$

the integral $= \int\limits_0^2 z \left(\int\limits_0^{\sqrt{4-4(z-1)^2}} rdr \right) dz = 2\int\limits_0^2 z(2z - z^2)\,dz = ...$

Or modified spherical coord.

9. Show that for any smooth conservative field $\mathbf{F}(x, y)$ and

any closed simple curve $c \subset \mathbb{R}^2$ $\oint\limits_c \mathbf{F} \cdot \mathbf{ds} = 0.$ [10]

using Green's Theorem

$\oint\limits_c \mathbf{F} \cdot \mathbf{ds} = \iint\limits_D \left[(F_2)_x - (F_1)_y \right] dxdy$ where $c = \partial D$

since the filed is conservative $(F_2)_x = (F_1)_y$ ...necessary cond.

so the integral is 0

Or

$\oint\limits_c \mathbf{F} \cdot \mathbf{ds} = f(B) - f(A) = 0$ since $A = B$ where $\mathbf{F} = \nabla f$.

10. *Bonus*

Calculate $\int\limits_0^1 \frac{x^a - 1}{\ln x} dx$ for $a > -1$. [10]

define $F(a) = \int\limits_0^1 \frac{x^a - 1}{\ln x} dx$ for $a > -1$ then $F(0) = 0$

$$F'(a) = \int\limits_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - 1}{\ln x}\right) dx = \int\limits_0^1 \frac{x^a \ln x}{\ln x} dx = \int\limits_0^1 x^a dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1} \text{ for } a + 1 > 0$$

then $F(a) = \int F'(a)da + c = \ln(a+1) + c$ but since $F(0) = 0$ also $c = 0$

and $\int\limits_0^1 \frac{x^a - 1}{\ln x} dx = \ln(a+1)$ for $a > -1$

to justify the differentiation estimate for $a \ge a_0 > -1$ ($\ln x < 0$)

$0 \le x^a = e^{a \ln x} \le e^{a_0 \ln x} = x^{a_0}$..... integrable majorant