THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 353 (L60)

SUMMER ,2000 TIME: 3 hours

1. Evaluate the integral $$\iint\limits_D \frac{x}{\sqrt{x^2+y^2}}dxdy$$

   where D is the region between two circles $x^2+y^2=4$ and $x^2+y^2=9$, on the right from the $y-$ axis.

   $D=\{4\le x^2+y^2\le 9, x\ge 0\}$
   use polar coord.then $D^*=\{4\le r^2\le 9, \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\}$

   $$\iint\limits_D \frac{x}{\sqrt{x^2+y^2}}dxdy=\iint\limits_{D^*}\frac{r\cos\theta}{\sqrt{r^2}}rdrd\theta=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\cdot\int\limits_2^3 rdr=$$

   $$=2\left[\sin\theta\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^2}{2}\right]_2^3=9-4=5.$$

2. Evaluate $$\iiint\limits_B\frac{zydV}{\sqrt{x^2+y^2+z^2}},\text{where } B=\{(x,y,z)\,;z\ge 0, y\ge 0, x^2+y^2+z^2\le 4\}.$$

   use spherical coord. $B^*=\left\{\phi\in\left[0,\frac{\pi}{2}\right],\theta\in[0,\pi],\rho^2\le 4\right\}$.

   $$\iiint\limits_B\frac{zydV}{\sqrt{x^2+y^2+z^2}}=\iiint\limits_{B^*}\frac{\rho\cos\phi\rho\sin\theta\sin\phi}{\sqrt{\rho^2}}\rho^2\sin\phi d\rho d\theta d\phi=$$

   $$=\int\limits_0^2\rho^3d\rho\cdot\int\limits_0^\pi\sin\theta d\theta\cdot\int\limits_0^{\frac{\pi}{2}}\cos\phi\sin^2\phi d\phi=\frac{2^4}{4}\left[-\cos\theta\right]_0^\pi\cdot\left[\frac{\sin^3\phi}{3}\right]_0^{\frac{\pi}{2}}=$$

   $$=4\cdot 2\cdot\frac{1}{3}=\frac{8}{3}.$$

3. Derive the formula for the surface area of a sphere $x^2+y^2+z^2=R^2$ for any $R>0.\,(SA=4\pi R^2)$.

   the surface S could be described as $z=\pm\sqrt{R^2-x^2-y^2}$
   for $(x,y)\in D=\{x^2+y^2\le R^2\}$

   then $\mathbf{n}=(\nabla z,-1)=\pm\left(\frac{-x}{\sqrt{R^2-x^2-y^2}},\frac{-y}{\sqrt{R^2-x^2-y^2}},-1\right)$

and $\|\mathbf{n}\| = \sqrt{\frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2} + 1} = \dfrac{R}{\sqrt{R^2-x^2-y^2}}$

so $SA = 2SA^+\,(z>0) = 2\iint\limits_{S^+} dS = 2\iint\limits_{D} \|\mathbf{n}\|\,dxdy =$

$= 2\iint\limits_{D} \dfrac{R}{\sqrt{R^2-x^2-y^2}}dxdy = 2R\iint\limits_{D^*} \dfrac{r}{\sqrt{R^2-r^2}}drd\theta\,(\text{polar}) =$

$= 2R\cdot 2\pi\left[-\sqrt{R^2-r^2}\right]_0^R = 4\pi R^2,$

where $D^* = \{0 \le r \le R, 0 \le \theta \le 2\pi\}$,

4. Use Green's Theorem to claculate $\displaystyle\int\limits_c \mathbf{F}\cdot\mathbf{ds}$

where $\mathbf{F} = (x^2y, xy^2)$ and the curve c is boundary of the ellipse $x^2+4y^2=1$, oriented counterclockwise.

for any smooth $\mathbf{F} = (F_1, F_2)$ by Green's Theorem

$\displaystyle\int\limits_c \mathbf{F}\cdot\mathbf{ds} = \iint\limits_D \left[(F_2)_x - (F_1)_y\right] dxdy$, where D is inside c

so for our field $(F_2)_x - (F_1)_y = y^2 - x^2$ and $D = \{\ x^2+4y^2 \le 1\}$

and $\displaystyle\int\limits_c \mathbf{F}\cdot\mathbf{ds} = \iint\limits_D \left[y^2 - x^2\right] dxdy$

by modified polar coord. $\quad x = r\cos\theta, y = \frac{1}{2}r\sin\theta$

we know that $x^2+4y^2 = r^2$ and $dxdy = \frac{1}{2}rdrd\theta$

thus

$$\int\limits_c \mathbf{F}\cdot\mathbf{ds} = \iint\limits_D \left[y^2-x^2\right]dxdy = \int_0^{2\pi}\int_0^1 \left[\tfrac{1}{4}r^2\sin^2\theta - r^2\cos^2\theta\right]\tfrac{1}{2}rdrd\theta =$$

$$= \tfrac{1}{8}\int_0^1 r^3dr\cdot\int_0^{2\pi}(\sin^2\theta - 4\cos^2\theta)d\theta = \tfrac{1}{8}\left[\tfrac{r^4}{4}\right]_0^1\cdot\int_0^{2\pi}(1-5\cos^2\theta)d\theta =$$

$$= \tfrac{1}{32}\int_0^{2\pi}(1 - 5\tfrac{1+\cos 2\theta}{2})d\theta = \tfrac{1}{32}\left[\tfrac{-3}{2}2\pi - 0\right] = -\frac{3}{32}\pi.$$

5. Show that for any smooth (i.e. with continuous second order partials) conservative vector field $\mathbf{F}$ of 3 variables $div\ \mathbf{F} = \triangle\Phi$,

where Φ is a potential of $\mathbf{F}$ and $\triangle$ is Laplace operator $\partial_{xx} + \partial_{yy} + \partial_{zz}$.

we know that $\mathbf{F} = \nabla\Phi = (\Phi_x, \Phi_y, \Phi_z)$ and $div\ \mathbf{F} = (F_1)_x + (F_2)_y + (F_3)_z$

together $div\ \mathbf{F} = (\Phi_x)_x + (\Phi_y)_y + (\Phi_z)_z = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = \triangle\Phi.$

6. Find the flux of $\mathbf{F} = (xyz, xy, z^2 + x^2)$ outward from the surface S -part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$ plane

   (a) including the bottom;

   (b) excluding the bottom.

$S_l = \{\ z = 4 - x^2 - y^2, (x, y) \in D\}$ ...lateral surface

and $S_b = \{\ z = 0, (x, y) \in D\}$ .....bottom

where $D = \{x^2 + y^2 \leq 4\}$

for a)

the surface is closed so we can use Gauss Theorem

flux $\iint\limits_S \mathbf{F} \cdot \mathbf{dS} = \iiint\limits_B div\mathbf{F}\ dxdydz \qquad$ where B is inside S

$B = \{\ 0 \leq z \leq 4 - x^2 - y^2, (x, y) \in D\}$

$div\mathbf{F} = (F_1)_x + (F_2)_y + (F_3)_z = (xyz)_x + (xy)_y + (z^2 + x^2)_z =$

$= yz + x + 2z$

$$\iint\limits_S \mathbf{F} \cdot \mathbf{dS} = \iiint\limits_B [x + (y + 2)\, z]\ dxdydz = 0 + 0 + \iint\limits_D (2) \left[\frac{z^2}{2}\right]_0^{4-x^2-y^2} dxdy$$

(since the integrand function is odd in x and set is symmetrical in x)

(since the integrand function is odd in y and set is symmetrical in y)

$$= \iint\limits_D (4 - x^2 - y^2)^2\, dxdy = (\text{ polar}) = \iint\limits_{D*} (4 - r^2)^2\, rdrd\theta =$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 (4 - r^2)^2\, rdr = 2\pi \left[\tfrac{-1}{6} (4 - r^2)^3\right]_0^2 = \tfrac{\pi}{3} \cdot 4^3 = \frac{64}{3}\pi.$$

for b)

S is not closed but we can use part a) since

$$\iint\limits_{S_l} \mathbf{F} \cdot \mathbf{dS} = \iint\limits_S \mathbf{F} \cdot \mathbf{dS} - \iint\limits_{S_b} \mathbf{F} \cdot \mathbf{dS} = \tfrac{64}{3}\pi - ?$$

so we have to calculate the flux through the bottom

$S_b = \{\ z = 0, (x, y) \in D\} \qquad \mathbf{n} = (0, 0, -1)$ and on S_b

$\mathbf{F} = (xyz, xy, z^2 + x^2)_{z=0} = (0, xy, x^2)$

$$\iint_{S_b} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\mathbf{n}\; dxdy = -\iint_D x^2 dxdy = -\int_0^{2\pi} \cos^2\theta d\theta \cdot \int_0^2 r^2 r dr =$$

$$= -\int_0^{2\pi} \tfrac{1+\cos 2\theta}{2} d\theta \cdot \left[\tfrac{r^4}{4}\right]_0^2 = -4\pi$$

therefore $\iint_{S_l} \mathbf{F}\cdot d\mathbf{S} = \frac{64}{3}\pi + 4\pi = \dfrac{76}{3}\pi.$

7. Evaluate $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{s}$

where $\mathbf{F} = (e^x - y^3, x^3 + e^y, e^z)$ and c is closed curve,

oriented counterclockwise $c = \{z = 2xy\} \cap \{x^2 + y^2 = 1\}$.

since the curve is closed we can use Stokes' Theorem

with $S = \{z = 2xy,$ for $x^2 + y^2 \leq 1\}$,

and upward $\mathbf{n} = (-\nabla z, 1) = (-2y, -2x, 1)$

we need $curl\mathbf{F} = \left\| \begin{array}{ccc} + & - & + \\ \partial_x & \partial_y & \partial_z \\ e^x - y^3 & x^3 + e^y & e^z \end{array} \right\| = (0, 0, 3x^2 + 3y^2)$

thus

$$\int_c \mathbf{F}\cdot d\mathbf{s} = \iint_S curl\mathbf{F}\cdot d\mathbf{S} = \int\int_{\{x^2+y^2\leq 1\}} (0, 0, 3x^2 + 3y^2)\cdot(-2y, -2x, 1)\, dxdy$$

$$= \int\int_{\{x^2+y^2\leq 1\}} (3x^2 + 3y^2)\, dxdy = 3\cdot 2\pi \int_0^1 r^3 dr = \frac{3}{2}\pi.$$

8. Evaluate $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{F} = (xy, y, z)$ and

$c = \{z = 2xy\} \cap \{x^2 + y^2 = 2\}$ between $A(-1, 1, -2)$ and $B(1, 1, 2)$.

the curve is not closed nor the field is conservative

so we have to find a parametrization of c

from the cylinder $x = \sqrt{2}\cos t,\ y = \sqrt{2}\sin t$ and from $z = 2xy$

we have $\mathbf{r}(t) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 4\cos t\sin t\right) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 2\sin 2t\right)$

now for A $t = \frac{3}{4}\pi$, and for B $t = \frac{\pi}{4}$

$\mathbf{r}'(t) = \left(-\sqrt{2}\sin t, \sqrt{2}\cos t, 4\cos 2t\right)$ and the field on c

$\mathbf{F} = (xy, y, z)_c = \left(2\sin t\cos t, \sqrt{2}\sin t, 2\sin 2t\right)$ then

$\mathbf{F}\cdot\mathbf{r}' = -2\sqrt{2}\sin^2 t\cos t + \sin 2t + 4\sin 4t$

$$\int_c \mathbf{F}\cdot\mathbf{ds} = \int_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \mathbf{F}\cdot\mathbf{r}' dt = -\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \left[-2\sqrt{2}\sin^2 t\cos t + \sin 2t + 4\sin 4t\right] dt =$$

$$= 2\sqrt{2}\left[\frac{\sin^3 t}{3}\right]_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} + \left[\frac{\cos 2t}{2}\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} - \left[\cos 4t\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} = 0 + 0 + 0 = 0.$$

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