MATH 353
Handout #2 Solutions

1. Find absolute extrema of $f(x,y) = \frac{1}{8}x^3 + y^3$ on the circle $x^2 + y^2 \leq 65$

   SOLUTION For 1)

   The function is cont.and the set is closed and bounded

   first C.P. inside $\nabla f = \mathbf{0}$ $\frac{3}{8}x^2 = 0 \qquad 3y^2 = 0 \qquad (0,0)$

   for C.P on the boundary use Langrange multiplier method

   where $g(x,y) = x^2 + y^2 = 65$

   solve $\nabla f = \lambda \nabla g$

   $\frac{3}{8}x^2 = \lambda 2x \to\to\to x = 0$ or $\frac{3}{8}x = 2\lambda$

   $3y^2 = \lambda 2y \to\to\to y = 0$ or $3y = 2\lambda$

   For $x = 0$ back to the circle $y = \pm\sqrt{65}$, similarly for $y = 0, x = \pm\sqrt{65}$

   for $xy \neq 0$

   $2\lambda = \frac{3}{8}x = 3y \Longrightarrow x = 8y$ back to the circle $\qquad 64\ y^2 + y^2 = 65$

   $y = \pm 1$ and $x = \pm 8$

   7 critical points:$(0,0)\,,\left(0, \pm\sqrt{65}\right), \left(\pm\sqrt{65}, 0\right), (\pm 8, \pm 1)$

   values of f : ..0...... $\pm\, 65\sqrt{65}$.... $\pm\, \frac{65}{8}\sqrt{65}$..... $\pm\, 65$

   So **max** is $65\sqrt{65}$ at $\left(0, \sqrt{65}\right)$ and **min** is $-65\sqrt{65}$ at $\left(0, -\sqrt{65}\right)$

2. Find the absolute extrema of $f(x,y) = x^2 + y^2$

   on the surface $S = \left\{\frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0.\right\}$.

   SOLUTION For 2)

   Notice that the roles of f and g are interchanged,

   and we need $x, y \geq 0$ to make S bounded.

   so $g(x,y) = \frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0$

   Solve $\nabla f = \lambda \nabla g$

   $2x = \lambda\frac{3}{8}x^2. \to\to\to x = 0$ or $\frac{16}{x} = 3\lambda$

   $2y = \lambda 3y^2 \to\to\to\to y = 0$ or $\frac{2}{y} = 3\lambda$

   For $x = 0$ back to $S \quad y = \sqrt[3]{65} = 4.02$,similarly, for $y = 0, x = 2\sqrt[3]{65}$

   for $xy \neq 0$

   $3\lambda = \dfrac{16}{x} = \dfrac{2}{y}$ so $x = 8y$ back to $S \quad 64y^3 + y^3 = 65$

   $y = 1, x = 8$

   3 critical points :$\left(0, \sqrt[3]{65}\right) \qquad \left(2\sqrt[3]{65}, 0\right) \qquad (8,1)$

   values of $f \qquad 65^{\frac{2}{3}} = 16.1 \qquad 4 \cdot 65^{\frac{2}{3}} = 64.66 \qquad 65$

   So **max** is 65 at $(8,1)$ and **min** is $65^{\frac{2}{3}}$ at $\left(0, \sqrt[3]{65}\right)$.

3. Find absolute maxim and minima of $f(x,y) = 2y^2 - x + x^2$
inside and on the triangle T with vertices $O(0,0)$, $A(1,1)$, $B(1,-1)$.

SOLUTION For 3)

Since the function is continuous and the set is closed and bounded we have to find all critical points inside and on the boundary,and check the values at those.

For critical points inside solve $\nabla f = \mathbf{0}$

$f_x = 2x - 1 = 0, f_y = 4y = 0$ so $x = \frac{1}{2}, y = 0$

Now, all critical points on the boundary

the boundary of T consists of 3 line segments

$B_1 = \{y = x, 0 \leq x \leq 1\}$ and $B_2 = \{y = -x, 0 \leq x \leq 1\}$

$f(x, \pm x) = 3x^2 - x = h(x)$,$h'(x) = 6x - 1 = 0$ for $x = \frac{1}{6} \to y = \pm\frac{1}{6}$

and the ends(corners)

$B_3 = \{x = 1, -1 \leq y \leq 1\}$ and f on B_3 is $f(1,y) = g(y) = 2y^2$

and $g'(x) = 4y = 0$ for $y = 0$, $x = 1$

Together all critical points in T and on ∂T are

$\left(\frac{1}{2}, 0\right), \left(\frac{1}{6}, \pm\frac{1}{6}\right), (0,0), (1, \pm 1), (1,0)$

Check the values of f

......$\frac{-1}{4}$.....$\frac{-1}{12}$ ....0..........2........0

So abs.max.value is 2 at the points $(1, \pm 1)$

and abs.min. value is $\frac{-1}{4}$ at the point $\left(\frac{1}{2}, 0\right)$.

4. Find the point on the plane $x - 2y - z = 3$ closest to the point $P(1,-1,2)$.
Justify!

For 4)

We are looking for minimum of the distance to P or the square of distance

$f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-2)^2$ and the constraint $g(x,y,z) = x - 2y - z = 3$

Solve $\nabla f = \lambda \nabla g$

$2(x-1) = \lambda$

$2(y+1) = -2\lambda$..................$\lambda = 2x - 2 = -y - 1 = 4 - 2z$

$2(z-2) = -\lambda$..........................$1 - 2x = y$..........$z = 3 - x$

back to the plane

$x - 2(1-2x) - (3-x) = 6x - 5 = 3 \to\to\to x = \frac{4}{3}, y = \frac{-5}{3}, z = \frac{5}{3}$

and the C.P. point is $\left(\frac{4}{3}, \frac{-5}{3}, \frac{5}{3}\right)$

To justify that we have found minimum

the set is not bounded but if we take a bounded part we have to have maximum and minimum

but the maximum will be on the boundary since the distance is increasing when we move far away

therefore the critical point must be minimum.

5. Find absolute maximum of $f(x,y,z) = xyz$

largest box with sides x, y, z on$\{(x,y,z)\,; 2xy+2xz+3yz = 144, x \geq 0, y \geq 0, z \geq 0\}$

(You may assume that there is an absolute maximum)

For 5)

$\nabla f = \lambda \nabla g$ where $g(x,y,z) = 2xy + 2xz + 3yz = 144$

$yz = \lambda(2y+2z)$ if $y=0$ or $z=0$ or $x=0$ then $f=0$

so we can assume that all $xyz \neq 0$

$xz = \lambda(2x+3z)$ $\frac{1}{\lambda} = \dfrac{2y+2z}{yz} = \dfrac{2x+3z}{xz} = \dfrac{2x+3y}{xy}$

$xy = \lambda(2x+3y)$

from the first equality (cancel z) ......$2yx + 2xz = 2xy + 3zy \rightarrow 2x = 3y$

from the second equality(cancel x)....$2xy + 3zy = 2xz + 3zy \rightarrow z = y$

back to the surface $3z^2 + 3z^2 + 3z^2 = 9z^2 = 144 \rightarrow z^2 = \frac{144}{9} = 16 \rightarrow z = \pm 4$

and critical points are $(\pm 6, \pm 4, \pm 4)$,values of f are ± 96

so maximum is 96 at the point $(6,4,4)$.

6. (a) Evaluate $\displaystyle\int_1^3 \left(\int_{-x}^{x^2} x e^{2y} dy \right) dx.$

(b) Switch the order of integration in the integral above and sketch the region D.

For 6a)

$$I = \int_1^3 \left(\int_{-x}^{x^2} x e^{2y} dy \right) dx = \int_1^3 x \left(\int_{-x}^{x^2} e^{2y} dy \right) dx. = \int_1^3 x \left(\left[\frac{e^{2y}}{2} \right]_{y=-x}^{y=x^2} \right) dx = \frac{1}{2} \int_1^3 x \left(e^{2x^2} - e^{-2x} \right) dx =$$

$$= \frac{1}{2} \int_1^3 x e^{2x^2} dx \,(\text{subst}) - \frac{1}{2} \int_1^3 x e^{-2x} dx \,(\text{by parts }) =$$

$$= \frac{1}{8} \left[e^{2x^2} \right]_1^3 - \frac{1}{2} \left[x \frac{e^{-2x}}{-2} \right]_1^3 - \frac{1}{4} \int_1^3 e^{-2x} dx = \frac{1}{8} \left[e^{18} - e^2 \right] + \frac{1}{4} \left[3e^{-6} - e^{-2} \right] + \frac{1}{8} \left[e^{-6} - e^{-2} \right] = ..$$

For b)

since we have three "left ends" we have to split the domain $D = D_1 \cup D_2 \cup D_3$where

$D_1 = \{(x,y)\,; -3 \leq y \leq -1, -y \leq x \leq 3\}$, $D_2 = \{(x,y)\,; -1 < y \leq 1, 1 \leq x \leq 3\}$

and $D_3 = \left\{(x,y)\,;1 < y \le 9, \sqrt{y} \le x \le 3\right\}$

and the integral

$$I = \int_{-3}^{-1}\left(\int_{-y}^{3} xe^{2y}dx\right)dy + \int_{-1}^{1}\left(\int_{1}^{3} xe^{2y}dx\right)dy + \int_{1}^{9}\left(\int_{\sqrt{y}}^{3} xe^{2y}dx\right)dy.$$

7. Evaluate $\iint_D \sqrt{2-x^2}dA$ where D is smaller region between $y = x^2$ and $x^2 + y^2 = 2$. and sketch the region

For 7)

find the intersection of the parabola $y = x^2$ and the circle $x^2 + y^2 = 2$

$y^2 + y - 2 = 0 \qquad (y-1)(y+2) = 0$

but y must be positive so $y = 1$ and $x = \pm 1$ and $D = \{-1 \le x \le 1\ x^2 \le y \le \sqrt{2-x^2}\}$

$$\iint_D \sqrt{2-x^2}dA = \int_{-1}^{1}\sqrt{2-x^2}\left(\int_{x^2}^{\sqrt{2-x^2}} dy\right)dx = \int_{-1}^{1}\sqrt{2-x^2}\left(\sqrt{2-x^2}-x^2\right)dx =$$

$$= \int_{-1}^{1}(2-x^2-x^2\sqrt{2-x^2})dx = (\text{even f.}) = 2\int_0^1 ...dx =$$

$$=2\cdot 2 - 2\left[\frac{x^3}{3}\right]_0^1 - 2\left(\text{Table } a = \sqrt{2}\,\right)\left[\frac{x}{8}\left(2x^2-2\right)\sqrt{2-x^2} + \frac{1}{2}\arcsin\frac{x}{\sqrt{2}}\right]_0^1 =$$

$$= 4 - \frac{2}{3} + 0 - \arcsin\frac{1}{\sqrt{2}} = \frac{10}{3} - \frac{\pi}{4}.$$

8. Switch the order of integration in the integral $\int_0^{\frac{\pi}{4}}\left(\int_0^{\tan x} f(x,y)dy\right)dx.$

For 8)

given $\quad 0 \le x \le \frac{\pi}{4}$ and $0 \le y \le \tan x$ sketch

$y = \tan x$ is equivalent to $\arctan y = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\frac{\pi}{4} = 1$

so $\quad 0 \le y \le 1$ and $\quad \arctan y \le x \le \frac{\pi}{4}$

and the integral

$$\int_0^{\frac{\pi}{4}}\left(\int_0^{\tan x} f(x,y)dy\right)dx = \int_0^1\left(\int_{\arctan y}^{\frac{\pi}{4}} f(x,y)dx\right)dy.$$

9. For $\iint_D \frac{1}{x^2+y}\,dA$ where D is the region between the x-axis and $y = 4 - x^2$ sketch the region D and set up BOTH iterated integrals and evaluate one of them. (Hint: $\lim\limits_{x\to 0^+} x\ln x = 0$).

For 9)

the region is above x-axis and below parabola $y = 4 - x^2$

so $\quad -2 \leq x \leq 2 \quad 0 \leq y \leq 4 - x^2$ or $0 \leq y \leq 4 \quad -\sqrt{4-y} \leq x \leq \sqrt{4-y}$

and

$$\iint_D \frac{1}{x^2+y}\, dA = \int_{-2}^{2} \left(\int_{0}^{4-x^2} \frac{1}{x^2+y} dy \right) dx = \int_{0}^{4} \left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2+y} dx \right) dy$$

evaluate the first ordered iterated integrals

$$\int_{-2}^{2} \left(\int_{0}^{4-x^2} \frac{1}{x^2+y} dy \right) dx = \int_{-2}^{2} \left[\ln\left(x^2+y\right)\right]_{y=0}^{y=4-x^2} dx = \int_{-2}^{2} \left(\ln 4 - \ln x^2\right) dx = 4\ln 4 - 2\int_{0}^{2} \ln x^2 dx =$$

$$= 4\ln 4 - 4\int_{0}^{2} \ln x dx = 8\ln 2 - 4\left[x\ln x - x\right]_0^2 = 8. \text{ (Otherwise } \ln x^2 = 2\ln|x|!\text{)},$$

using the limit of $x\ln x \to 0$ as $x \to 0^+$.

The other way

$$\int_{0}^{4} \left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2+y} dx \right) dy = \int_{0}^{4} \left(\left[\frac{1}{\sqrt{y}} \arctan \frac{x}{\sqrt{y}}\right]_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}} \right) dy = \text{....harder}$$

10. Calculate the volume of the solid below the surface $z = e^{(y-1)^2}$ and above the triangle T with vertices $A(-1,0), B(0,1), C(2,0)$ with vertical sides.

for 10)

$V = \iint_T e^{(y-1)^2}\, dxdy \qquad$ it is easier to slice the triangle horizontally

$0 \leq y \leq 1 \qquad line_{AB} \leq x \leq line_{BC}$

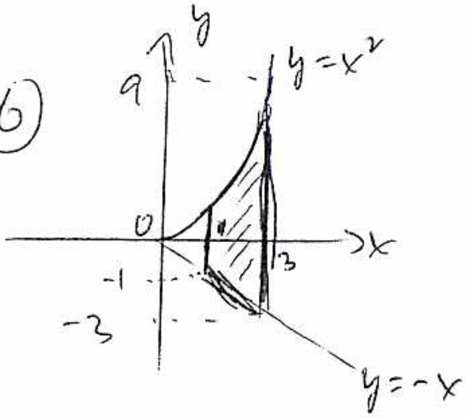
where line AB $\qquad y = x + 1$ or $x = y - 1$

lineBC $\qquad y = 1 - \frac{1}{2}x$ or $x = 2 - 2y$

so $\quad V = \int_{0}^{1} \left(e^{(y-1)^2} \int_{y-1}^{2-2y} dx \right) dy = \int_{0}^{1} e^{(y-1)^2} (3 - 3y)\, dy = -\frac{3}{2} \int_{1}^{0} e^u du = \frac{3}{2}[e - 1]$

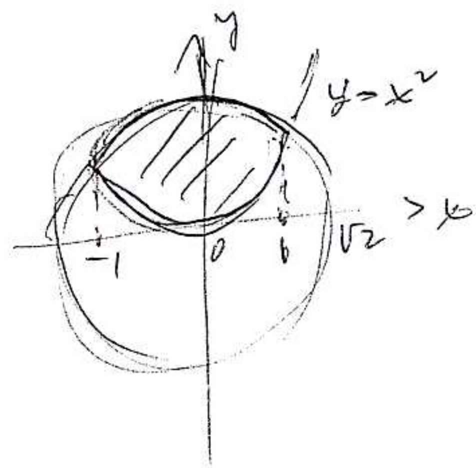
by subst. $u = (y-1)^2$

(6)



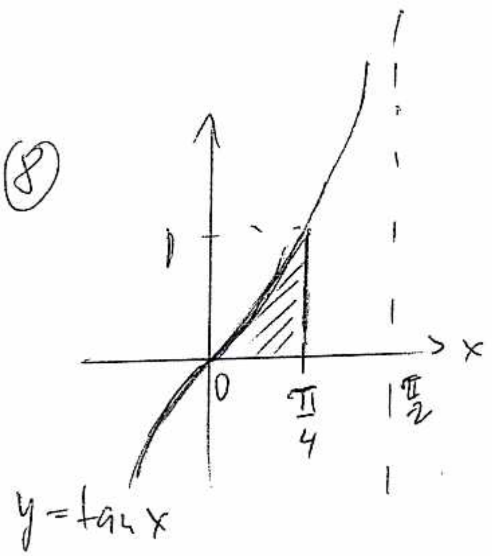
y
9
$y=x^2$
0
x
1
3
-1
-3
$y=-x$

(7)



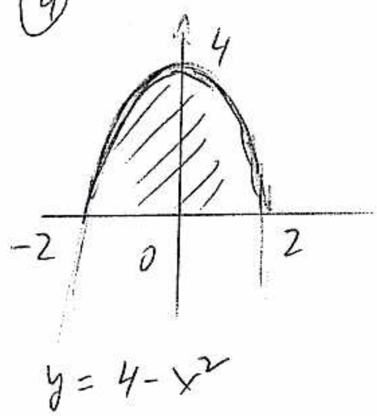
y
$y=x^2$
x
-1
0
1
$\sqrt{2}$

(8)



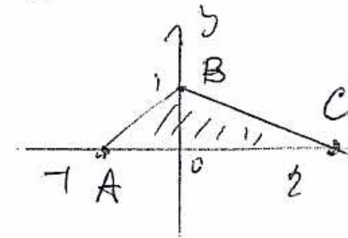
1
x
0
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$y=\tan x$

(9)



4
-2
0
2
$y=4-x^2$

(10)



y
1
B
C
-1
A
0
2