MATH 353
Handout #5 Solution

For1)
let's find the intersection of two given surfaces
$z = x^2 + y^2 \to z + z^2 = 2$ so $z^2 + z - 2 = 0$ has positive sol. $z = 1$
and we can describe the surface
$S = \left\{ z = \sqrt{2 - x^2 - y^2}, (x, y) \in D \right\}$where $D = \{x^2 + y^2 \leq 1\}$
and normal $\mathbf{n} = \pm (\nabla z, -1) = \pm \left(\frac{x}{\sqrt{2-x^2-y^2}}, \frac{y}{\sqrt{2-x^2-y^2}}, 1 \right)$
and $\|\mathbf{n}\| = \sqrt{\frac{2}{2-x^2-y^2}}$,finally the surface area

$$A = \iint_S dS = \iint_D \|\mathbf{n}\| \, dxdy = \sqrt{2} \iint_D \frac{dxdy}{\sqrt{2 - x^2 - y^2}} = (\text{ polar})$$

$$= 2\pi\sqrt{2} \int_0^1 \frac{rdr}{\sqrt{2 - r^2}} = 2\sqrt{2}\pi \left[-\sqrt{2 - r^2} \right]_0^1 = 2\pi \left[2 - \sqrt{2} \right].$$

For 2)
the field on the lateral part of S_l is $\mathbf{F} = (1, 1, 16z)$
but since this part is vertical $x^2 + y^2 = 4, z \in [0, 3]$
we cannot describe it as $z = f(x, y)$ so parametrize $\mathbf{r}(u, v)$:
$x = 2\cos u \quad y = 2\sin u \quad z = v \quad u \in [0, 2\pi], v \in [0, 3]$
$\frac{\partial \mathbf{r}}{\partial u} = (-2\sin u, 2\cos u, 0) \qquad \frac{\partial \mathbf{r}}{\partial v} = (0, 0, 1)$ and the normal vector
$\mathbf{n} = \pm \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) = (2\cos u, 2\sin u, 0)$ ($\|\mathbf{n}\| = 2$)
for outward take +
$\mathbf{F} \cdot \mathbf{n} = 2(\cos u + \sin u)$

$$\iint_{S_l} \mathbf{F} \cdot \mathbf{dS} = \mathbf{2} \int_0^{2\pi} \int_0^3 (\cos u + \sin u) \, dvdu = 0 \text{ because of the periodicity.}$$

The top could be described as $z = 3$
$(x, y) \in D = \{x^2 + y^2 \leq 4\}$ $\mathbf{n} = (0, 0, 1)$ upward
the field on the top is $\mathbf{F} = (1, 1, 3(x^2 + y^2)^2)$ and $\mathbf{F} \cdot \mathbf{n} = 3(x^2 + y^2)^2$
the flux comming out from the top is

$$\iint_{top} \mathbf{F} \cdot \mathbf{dS} = \iint_D 3(x^2 + y^2)^2 dxdy = (\text{ polar}) = 6\pi \int_0^2 r^5 dr = 3 \cdot 2^6 \pi.$$

The bottom could be described as $z = 0$
$(x, y) \in D = \{x^2 + y^2 \leq 4\}$ $\mathbf{n} = (0, 0, -1)$ downward
the field on the bottom is $\mathbf{F} = (1, 1, 0)$ and $\mathbf{F} \cdot \mathbf{n} = 0$.
Therefore the only contribution to the flux is out from the top and

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = 192\pi.$$

For 3 a)
the surface S is vertical so parametrize:
$x = 2\cos u, y = 2\sin u, u \in \left[0, \frac{\pi}{2} \right]$, and $0 \leq z \leq 5 - 2x - y$ (below the plane)
necessary condition $2x + y \leq 5$ is satisfied inside the circle

$\mathbf{r}(u,v) = (2\cos u, 2\sin u, v)$ and $D = \left\{u \in \left[0, \frac{\pi}{2}\right], 0 \le v \le 5 - 4\cos u - 2\sin u\right\}$

$\dfrac{\partial \mathbf{r}}{\partial u} = (-2\sin u, 2\cos u, 0)$, $\dfrac{\partial \mathbf{r}}{\partial v} = (0,0,1)$ so $\mathbf{n} = \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} = (2\cos u, 2\sin u, 0)$

thus $\|\mathbf{n}\| = 2$ and $SA = \iint_S dS = \iint_D \|\mathbf{n}\|\, dudv = 2\int_0^{\frac{\pi}{2}} \left(\int_0^{5-4\cos u - 2\sin u} dv \right) du =$

$= 2\int_0^{\frac{\pi}{2}} (5 - 4\cos u - 2\sin u) du = 5\pi + [-8\sin u + 4\cos u]_0^{\frac{\pi}{2}} = 5\pi - 8 - 4 = 5\pi - 12.$

For 3 b)

this time S is given as $z = 5 - 2x - y$, $(x,y) \in D = \{x^2 + y^2 \le 4\}$

then $\mathbf{n} = (-2,-1,-1)$ and $\|\mathbf{n}\| = \sqrt{6}$

$SA = \iint_S dS = \iint_D \|\mathbf{n}\|\, dxdy = \sqrt{6}\text{area of } D = 4\sqrt{6}\pi.$

For 4)

$S : z = \sqrt{4 - y^2}, (x,y) \in D = \{x^2 + y^2 \le 4\}$ since $z > 0$

normal vector $\mathbf{n} = (-\nabla z, 1) = \left(0, \dfrac{y}{\sqrt{4-y^2}}, 1\right)$

since upward means positive z-coordinate

$\mathbf{F} = (x^2yz, y, xz)$ on S $\quad \mathbf{F} = \left(..., y, x\sqrt{4-y^2}\right)$ and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\mathbf{n}\; dxdy = \iint_D \left[\frac{y^2}{\sqrt{4-y^2}} + x\sqrt{4-y^2}\right] dxdy =$$

$$= \int_{-2}^{2} \left(\frac{y^2}{\sqrt{4-y^2}} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\right) dy + \int_{-2}^{2} \sqrt{4-y^2} \left(\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x dx \right) dy =$$

$$= 2\int_0^2 \frac{y^2}{\sqrt{4-y^2}} \cdot 2\sqrt{4-y^2} dy + 0(x \text{ is an odd f}) = 4\int_0^2 y^2 dy = \tfrac{4}{3} \cdot 8 = \frac{32}{3}.$$

For 5)

$S : z = \frac{x^2}{2}, (x,y) \in D = \{x^2 + y^2 \le 1, x > 0, y < 0\}$

normal vector $\mathbf{n} = (\nabla z, -1) = (x, 0, -1)$ and $\|\mathbf{n}\| = \sqrt{x^2+1}$

and

$$\int_S\int zx\; dS = \int_D\int \tfrac{x^2}{2} \cdot x\sqrt{x^2+1}\; dxdy = \tfrac{1}{2}\int_0^1 x^3\sqrt{x^2+1}\left(\int_{-\sqrt{1-x^2}}^{0} dy\right)dx =$$

$$= \tfrac{1}{2}\int_0^1 x^3\sqrt{x^2+1}\sqrt{1-x^2}dx = \tfrac{1}{2}\int_0^1 x^3\sqrt{1-x^4}dx =$$

$(u = 1 - x^4, du = -4x^3dx)$ $\quad = \tfrac{1}{8}\int_0^1 \sqrt{u}du = \tfrac{1}{8}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \dfrac{1}{12}.$

For 6)

Evaluate where S : $z = 2 - x - y$ for $(x,y) \in D = \{x^2 + 2y^2 \le 1\}$

since for $z = 0$ the line $x + y = 2$ is outside the ellipse

so $\mathbf{n} = (\nabla z, -1) = (-1,-1,-1)$ and $\|\mathbf{n}\| = \sqrt{3}$

and $\iint\limits_S x^2 dS = \int\limits_D\int x^2\sqrt{3}\ dxdy$ (modified polar . or cartesian coord.)

$x = r\cos\theta \qquad y = \frac{1}{\sqrt{2}} r\sin\theta \qquad dxdy = \frac{1}{\sqrt{2}} r dr d\theta \qquad x^2 + 2y^2 = r^2$

then $D^* = \{0 \le r \le 1, 0 \le \theta \le 2\pi\}$ and the integral

$$= \sqrt{\tfrac{3}{2}} \int_0^{2\pi}\int_0^1 r^2\cos^2\theta\ r d\theta dr = \sqrt{\tfrac{3}{2}} \left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{\sqrt{3}\pi}{4\sqrt{2}}$$

OR $\quad x \in [-1,1]$ and $y \in \left[-\frac{1}{\sqrt{2}}\sqrt{1-x^2}, \frac{1}{\sqrt{2}}\sqrt{1-x^2}\right]$.