The University of Calgary
Faculty of Science
Department of Mathematics and Statistics
MATH 353
Midterm Supplement

1. Describe in the xyz - space the following sets

   (a) where $\rho = x$ (ρ is from spherical coord.);

   (b) where $\rho = -2y$.

   for 1a)

   $\sqrt{x^2+y^2+z^2} = x$ so $x \geq 0$ and $x^2+y^2+z^2 = x^2$, $y^2+z^2 = 0$

   thus $y = z = 0$ and any point $(x, 0, 0)$ for $x \geq 0$ - half of the x-axis

   for b)

   $\sqrt{x^2+y^2+z^2} = -2y$ so $y \leq 0$ and $x^2+y^2+z^2 = 4y^2$

   $x^2+z^2 = 3y^2$,one part cone around the y - axis $y = -\dfrac{\sqrt{x^2+z^2}}{\sqrt{3}}$.

2. Set up the integral $\iiint\limits_B z\, dxdydz$ where B is the region in the first octant below the plane $z = 2$

   and above the plane $3x + 2y - 6z = 0$ as iterated integrals

   (a) with $\int dz$ inside ;

   (b) with $\int dz$ outside

   then evaluate only once.

   for 2a)

   first $B = \{(x, y, z)\,; x \geq 0, y \geq 0, z \geq 0, \dfrac{3x+2y}{6} \leq z \leq 2\}$

   "bottom" must be below "top" $\dfrac{3x+2y}{6} \leq 2$ and we get

   $D = \{(x, y)\,; x \geq 0, y \geq 0, 3x + 2y \leq 12\}$ a triangle

   so $$\iiint\limits_B z\, dxdydz = \iint\limits_D \left(\int_{\frac{3x+2y}{6}}^{2} z\, dz \right) dxdy = \int_0^4 \left[\int_0^{\frac{12-3x}{2}} \left(\int_{\frac{3x+2y}{6}}^{2} z\, dz \right) dy \right] dx = ..$$

   for b)

   $0 \leq z \leq 2$ and for a fixed z

   $D_z = \{(x, y)\,; x \geq 0, y \geq 0, 3x + 2y \leq 6z\}$

   triangles with the base $[0, 2z]$ and the height $[0, 3z]$ and the area $3z^2$

   $$\iiint\limits_B z\, dxdydz = \int_0^2 z \left(\iint\limits_{D_z} dxdy \right) dz = \int_0^2 z\,(\text{area of } D_z)\, dz = 3\int_0^2 z^3 dz = 12$$

OR

$$\iiint\limits_B z\ dxdydz = \int_0^2 z \left(\int_0^{2z} (\int_0^{\frac{6z-3x}{2}} dy)dx \right) dz = ...$$

3. Set up the integral $\displaystyle\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}}$

where $B = \{(x,y,z)\,;x^2+y^2+z^2 \le 4\ ;x^2+y^2 \ge 3, x \ge 0, y \ge 0\}$
as iterated integrals

(a) in spherical coordinates;
(b) in cylindrical coordinates,
then evaluate only once.

for 3 a)

the solid B is outside the cylinder and inside the sphere

$x \ge 0, y \ge 0$ implies $\theta \in [0, \frac{\pi}{2}]\,, x^2+y^2+z^2 \le 4 \to 0 \le \rho \le 2$

and $x^2+y^2 \ge 3$ implies $\rho \sin\phi \ge \sqrt{3}$ $\qquad \dfrac{\sqrt{3}}{\sin\phi} \le \rho \le 2$

so necessarily $\qquad \frac{\sqrt{3}}{2} \le \sin\phi \qquad \phi \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$

$B^* = \{\ \theta \in [0, \frac{\pi}{2}]\,, \phi \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right], \dfrac{\sqrt{3}}{\sin\phi} \le \rho \le 2\}$

as iterated integrals

$$\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint\limits_{B^*} \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{\rho} = \frac{\pi}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left(\int_{\frac{\sqrt{3}}{\sin\phi}}^{2} \rho\ d\rho \right) d\phi =$$

$$= \frac{\pi}{4} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left(4 - \tfrac{3}{\sin^2\phi}\right) d\phi = \pi \left[-\cos\phi\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} - \frac{3\pi}{4} \left[\ln\left|\csc\phi - \cot\phi\right|\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} =$$

$\left(\cos\frac{\pi}{3} = \frac{1}{2}, \cos\frac{2\pi}{3} = -\frac{1}{2}, \sin\left(\text{both}\right) = \frac{\sqrt{3}}{2}, \cot = \pm\frac{1}{\sqrt{3}}\right)$

$= \pi - \frac{3\pi}{2} \ln\sqrt{3} = \pi\left(1 - \frac{3}{4}\ln 3\right)$

for b)

the solid B is outside the cylinder and inside the sphere

$x \ge 0, y \ge 0$ implies $\theta \in [0, \frac{\pi}{2}]\,, x^2+y^2+z^2 \le 4 \to r^2+z^2 \le 4$

and $x^2+y^2 \ge 3$ implies $\qquad r \ge \sqrt{3}$

$B^* = \{\ \theta \in [0, \frac{\pi}{2}]\,, r \ge \sqrt{3}, r^2+z^2 \le 4\}$

$$\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint\limits_{B^*} \frac{r dr dz d\theta}{\sqrt{r^2+z^2}} = \frac{\pi}{2} \int_{\sqrt{3}}^{2} r \left(\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \frac{dz}{\sqrt{r^2+z^2}} \right) dr =$$

$$\text{OR} = \frac{\pi}{2}\int_{-1}^{1}\left(\int_{\sqrt{3}}^{\sqrt{4-z^2}} \frac{r\ dr}{\sqrt{r^2+z^2}}\right) dz = \pi\int_0^1 \left(\sqrt{r^2+z^2}\right)\Big|_{\sqrt{3}}^{\sqrt{4-z^2}} dz = \pi\int_0^1 \left(2-\sqrt{3+z^2}\right) dz =$$

$$= 2\pi - \frac{\pi}{2}\left[z\sqrt{3+z^2} + 3\ln\left|z+\sqrt{3+z^2}\right|\right]_0^1 = 2\pi - \frac{\pi}{2}\left[2+3\ln 3 - 3\ln\sqrt{3}\right] =$$

$$= \pi\left[1-\frac{3}{4}\ln 3\right].$$

4. Evaluate $\iiint\limits_B \frac{dxdydz}{z-6}$ where B is the solid bounded by planes $x=0, y=0, z=0,$ and $3x+3y+z=6$. HINT: Iterate in such a way that $\int dz$ is outside!

for 4)

the solid is a tetrahedron

in the first octant under the plane $3x+3y+z=6$

so $\quad B = \{x \geq 0, y \geq 0, z \geq 0\ ,\ 3x+3y+z \leq 6\}$

for $z=0$ we have a triangle $(0,0), (2,0), (0,2)$ also D

for $z=6 \qquad x=y=0$

for $z \in (0,6) \qquad$ we have a smaller triangle

under the line $3x+3y=6-z$ with vertices $(0,0), \left(0, \frac{6-z}{3}\right), \left(\frac{6-z}{3}, 0\right)$

$D_z = \{x \geq 0, y \geq 0, x+y \leq \frac{6-z}{3}\}$ ...triangle with area $A_z = \frac{1}{2}\left(\frac{6-z}{3}\right)^2$

$$\iiint\limits_B \frac{dxdydz}{z-6} = \int_0^6 \frac{1}{z-6}\left(\iint\limits_{D_z} dxdy\right) dz = \int_0^6 \frac{1}{z-6}\cdot A_z\ dz = \frac{1}{18}\int_0^6 (z-6)\ dz =$$

$$= \frac{1}{36}\left[(z-6)^2\right]_0^6 = \frac{-36}{36} = -1.$$

5. Set the integral $\iiint\limits_B z\ dxdydz$ where $B = \{x^2+y^2+z^2 \leq 2, z \geq 0, y \geq 0, x^2+y^2 \geq z\}$ as iterated integrals in both

   (a) cylindrical and

   (b) spherical coordinates $\qquad$ then evaluate only one of the above .

for 5a)

the set is inside the sphere with radius $\sqrt{2}$

below the paraboloid $z = x^2+y^2$, and above the xy-plane $z=0$

$B^* = \{r^2+z^2 \leq 2, z \geq 0, r \geq 0, \theta \in [0,\pi], r^2 \geq z\}$

so $I = \iiint\limits_B z\ dxdydz = \iiint\limits_{B^*} z\ rdrd\theta dz = \pi \iint\limits_D rz\ drdz$

where $D = \{r^2+z^2 \leq 2, z \geq 0, r \geq 0, r^2 \geq z\}$

intersection of the circle and parabola is at $z=r=1$

it is easier to slice it horizontally in $zr-$ plane so $\qquad z \in [0,1]$ and $r \geq \sqrt{z}$, $r \leq \sqrt{2-z^2}$

the integral $I = \pi \int\limits_0^1 z \left(\int\limits_{\sqrt{z}}^{\sqrt{2-z^2}} r dr \right) dz = \pi \int\limits_0^1 z \left(\left[\frac{r^2}{2} \right]_{\sqrt{z}}^{\sqrt{2-z^2}} \right) dz =$

$= \frac{1}{2}\pi \int\limits_0^1 z \left(\left[2 - z^2 - z \right] \right) dz = \frac{\pi}{2} \left[z^2 - \frac{z^4}{4} - \frac{z^3}{3} \right]_0^1 = \frac{\pi}{2} \cdot \left(1 - \frac{7}{12} \right) = \frac{5}{24}\pi.$

For b)

since $x^2 + y^2 = \rho^2 \sin\phi$

$B^* = \left\{ \rho \leq \sqrt{2}, \theta \in [0,\pi], \phi \in \left[0, \frac{\pi}{2}\right], \rho \geq \frac{\cos\phi}{\sin^2\phi} \right\}$

since $x^2 + y^2 = \rho^2 \sin^2\phi$ and $z = \rho\cos\phi$

but necessarily $\qquad \sqrt{2} \geq \dfrac{\cos\phi}{\sin^2\phi} \qquad \to \phi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

thus

$I = \iiint\limits_B z \; dxdydz = \iiint\limits_{B^*} \rho^3 \cos\phi \sin\phi \; d\rho d\phi d\theta =$

$= \int\limits_0^\pi d\theta \cdot \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\cos\phi \sin\phi \int\limits_{\frac{\cos\phi}{\sin^2\phi}}^{\sqrt{2}} \rho^3 d\rho \right) d\phi ...$

6. Evaluate $\iiint\limits_B \sqrt{x^2 + 2y^2} \; dxdydz$ where B is the solid bounded by surfaces $z = x^2 + y^2$and $z = 4 - x^2 - 3y^2$.

For 6)

$B = \left\{ x^2 + y^2 \leq z \leq 4 - x^2 - 3y^2 \right\}$ between two paraboloids

necessarily $x^2 + y^2 \leq 4 - x^2 - 3y^2$

so $2x^2 + 4y^2 \leq 4$ that is $(x,y) \in D_0 = \left\{ x^2 + 2y^2 \leq 2 \right\}$

and

$\iiint\limits_B \sqrt{x^2 + 2y^2} dxdydz = \iint\limits_{D_0} \sqrt{x^2 + 2y^2} \left(\int\limits_{x^2+y^2}^{4-x^2-3y^2} dz \right) dxdy =$

$= \iint\limits_{D_0} \sqrt{x^2 + 2y^2} \left(4 - 2\left(x^2 + 2y^2 \right) \right) dxdy$

use modified polar coord. to evaluate $\qquad x = r\cos\theta, y = \frac{1}{\sqrt{2}} r \sin\theta$

then $\qquad x^2 + 2y^2 = r^2 \qquad dxdy = \frac{1}{\sqrt{2}} r dr d\theta$

so D_0 transforms into $D^* = \left\{ 0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi \right\}$

and the integral into

$= \int\limits_0^{2\pi} d\theta \int\limits_0^{\sqrt{2}} \left(4r - 2r^3 \right) \frac{r}{\sqrt{2}} dr = \sqrt{2}\pi \left[\frac{4r^3}{3} - \frac{2r^5}{5} \right]_0^{\sqrt{2}} = \frac{32}{15}\pi.$

7. For the solid B in the first octant bounded by the coordinate planes, the plane $y+z=2$ and the surface $x=4-y^2$.
set up two different ways of integration of $\iiint\limits_B f dxdydz$

(a) (double first,then single) $\iint\limits_{D_o} \left(\int f dz \right) dxdy$; sketch D_0;

(b) (single first,then double) $\int\limits_a^b \left(\iint\limits_{D_z} f dxdy \right) dz$; sketch D_z.

For 7)

$B = \left\{ x \geq 0, y \geq 0, z \geq 0, z \leq 2-y, x \leq 4-y^2 \left(\text{ or } y \leq \sqrt{4-x} \right) \right\}$

so

$D_0 = \left\{ x \geq 0, y \geq 0, x \leq 4-y^2 \left(\text{ or } y \leq \sqrt{4-x} \right) \right\}$

and for a)

$$\iiint\limits_B f dxdydz = \iint\limits_{D_o} \left(\int_0^{2-y} f dz \right) dxdy$$

for b)

$0 \leq z \leq 2$ and for a fixed z $\quad D_z = \{0 \leq y \leq 2-z, 0 \leq x \leq 4-y^2\}$

$$\iiint\limits_B f dxdydz = \int\limits_0^2 \left(\iint\limits_{D_z} f dxdy \right) dz$$