The University of Calgary
Faculty of Science
Department of Mathematics and Statistics
MATH 353 -01/02
Midterm Test

Time:90 minutes **Solution** **Winter 2006**

1. For $f(x,y,z) = xy - z^2 + x$ find the absolute extrema on the set $B = \{(x,y,z,)\,;x^2+y^2+z^2 \le 1\}$..

C.P. inside $\nabla f = \mathbf{0}$

$y+1=0 \qquad x=0 \qquad -2z=0 \qquad (0,-1,0)$

C.P. on the boundary $\nabla f = \lambda \nabla g \qquad g(x,y,z) = x^2+y^2+z^2 = 1$

$y+1 = \lambda 2x \qquad x = \lambda 2y \qquad -2z = \lambda 2z$

from the last equation case 1.$z=0$;case 2.$z \ne 0$ so $\lambda = -1$

for 1. $xy \ne 0; \dfrac{x}{2y} = \lambda = \dfrac{y+1}{2x} \qquad x^2 = y^2 + y$ together with $x^2+y^2=1$

$2y^2 + y - 1 = 0 \qquad (2y-1)(y+1) = 0$ for $y=-1$ as above

for $y = \frac{1}{2} \qquad x^2 = \frac{3}{4}$so $x = \pm\frac{\sqrt{3}}{2}$ finally $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$

for 2. $\qquad \lambda = -1 \to y+1 = -2x \qquad x = -2y$ so $y+1 = 4y$

thus $y = \frac{1}{3} \qquad x = \frac{-2}{3}$ and $z^2 = 1 - \frac{1}{9} - \frac{4}{9} \qquad \left(\frac{-2}{3}, \frac{1}{3}, \pm\frac{2}{3}\right)$

5 C.P.,now values

$f(0,-1,0) = 0 \qquad f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right) = \frac{3\sqrt{3}}{4} = 1.3$ max

$f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right) = -\frac{3\sqrt{3}}{4} = -1.3; f\left(\frac{-2}{3}, \frac{1}{3}, \pm\frac{2}{3}\right) = -\frac{4}{3} = -1.33$ min

2. Evaluate $\displaystyle\iint_D \frac{y}{x+y^2}\,dxdy$ where D is the region in the first quadrant between $x=4$ and $y=\sqrt{x}$.

$$\iint_D \frac{y}{x+y^2}\,dxdy = \int_0^4 \left(\int_0^{\sqrt{x}} \frac{y}{x+y^2}dy\right) dx = \frac{1}{2}\int_0^4 \left[\ln\left(x+y^2\right)\right]_{y=0}^{y=\sqrt{x}} dx =$$

$$= \frac{1}{2}\int_0^4 \left[\ln(2x) - \ln x\right] dx = \frac{1}{2}\int_0^4 \ln\left(\frac{2x}{x}\right) dx = \frac{1}{2}\ln 2 \cdot 4 = 2\ln 2$$

OR $$\iint_D \frac{y}{x+y^2}\,dxdy = \int_0^2 y\left(\int_{y^2}^4 \frac{1}{x+y^2}dx\right) dy = \int_0^2 y\left[\ln\left(x+y^2\right)\right]_{y^2}^4 dy =$$

$$= \int_0^2 y\left[\ln\left(4+y^2\right) - \ln 2y^2\right] dy \text{ (subst. } y^2 = u) \ldots$$

3. Describe/sketch the set S in the coordinates x, y, z

   (a) if $S = \{\rho = 2r\}$
   where ρ is from spherical , r from cylindrical coordinates;

   (b) if $S = \{r = 2, \theta = \frac{\pi}{4}, z \text{ any}\}$ is given in cylindrical coordinates.

   for a)

   $\rho = 2r \to x^2 + y^2 + z^2 = 4x^2 + 4y^2 \qquad z^2 = 3(x^2 + y^2); z = \pm\sqrt{3(x^2 + y^2)}$

   two part cone

   for b) $r = 2, \theta = \frac{\pi}{4}$ gives the point $\left(\sqrt{2}, \sqrt{2}\right)$together with z any

   vertical line $\left(\sqrt{2}, \sqrt{2}, z\right)$

4. Evaluate the integral $\displaystyle\iiint_B \frac{dxdydz}{\sqrt{z}}$

   where B is the region in the first octant below $z = \sqrt{x + y}$ and above the triangle with vertices $(0, 0)$ $(1, 0)$ and $(0, 1)$.

   $B = \{0 \le z \le \sqrt{x + y}, (x, y) \in T\}$ where $T = \{x \ge 0, y \ge 0, x + y \le 1\}$

   so $\displaystyle\iiint_B \frac{dxdydz}{\sqrt{z}} = \iint_T \left(\int_0^{\sqrt{x+y}} \frac{1}{\sqrt{z}} dz\right) dxdy = 2 \iint_T [\sqrt{z}]_0^{\sqrt{x+y}} dxdy =$

   $\displaystyle= 2 \iint_T (x + y)^{\frac{1}{4}} dxdy = 2 \int_0^1 \left(\int_0^{1-x} (x + y)^{\frac{1}{4}} dy\right) dx = \tfrac{8}{5} \int_0^1 \left[(x + y)^{\frac{5}{4}}\right]_{y=0}^{y=1-x} dx =$

   $\displaystyle= \tfrac{8}{5} \int_0^1 \left[1 - x^{\frac{5}{4}}\right] dx = \tfrac{8}{5}(1 - \tfrac{4}{9}) = \frac{8}{9}$

   OR $\displaystyle\iiint_B \frac{dxdydz}{\sqrt{z}} = \int_0^1 \frac{1}{\sqrt{z}} \left(\iint_{T_z} dxdy\right) dz$ where $T_z = \{x \ge 0, y \ge 0, z^2 \le x + y \le 1\}$

   area of T_z is the diffrence between two triangles so $A_z = \frac{1}{2}\left(1 - z^4\right)$

   $\displaystyle\int_0^1 \frac{1}{\sqrt{z}} (A_z)\, dz = \tfrac{1}{2} \int_0^1 \left(z^{-\frac{1}{2}} - z^{\frac{7}{2}}\right) dz = \left[\sqrt{z} - \tfrac{1}{9} z^{\frac{9}{2}}\right]_0^1 = \frac{8}{9}.$

5. Evaluate $\displaystyle\iiint_B \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}}$where $B = \{(x, y, z)\,; x^2 + y^2 + z^2 \le 2, z \ge 1, x \ge 0\}$.

   spherical coord.

   $\displaystyle\iiint_B \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}} = \iiint_{B^*} \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{\rho}$ where $B^* = \{\rho^2 \le 2, \rho\cos\phi \ge 1, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$

   together $\displaystyle\frac{1}{\cos\phi} \le \rho \le \sqrt{2}$ necessarily $\frac{1}{\sqrt{2}} \le \cos\phi$ so $\phi \in \left[0, \frac{\pi}{4}\right]$

$$\iiint\limits_B \frac{dxdydz}{\sqrt{x^2+y^2+z^2}} = \iiint\limits_{B^*} \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{\rho} = \left(\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\right) \int\limits_0^{\frac{\pi}{4}} \sin\phi \left(\int\limits_{\frac{1}{\cos\phi}}^{\sqrt{2}} \rho d\rho\right) d\phi =$$

$$= \tfrac{\pi}{2} \int\limits_0^{\frac{\pi}{4}} \sin\phi \left(2 - \tfrac{1}{\cos^2\phi}\right) d\phi = \pi \left[-\cos\phi\right]_0^{\frac{\pi}{4}} - \tfrac{\pi}{2} \int\limits_0^{\frac{\pi}{4}} \sin\phi \left(\tfrac{1}{\cos^2\phi}\right) d\phi = \tfrac{\pi}{2}\{\left(2-\sqrt{2}\right) - \left[\tfrac{1}{\cos\phi}\right]_0^{\frac{\pi}{4}}\}$$

$$= \tfrac{\pi}{2}\left[2-\sqrt{2}-\sqrt{2}+1\right] = \pi\left(\tfrac{3}{2}-\sqrt{2}\right).$$

6. Evaluate $\iiint\limits_B z\sqrt{x^2+y^2}dxdydz$

where $B = \{(x,y,z)\,; 2x^2+2y^2+z^2 \leq 3, z \geq \sqrt{x^2+y^2}, y \geq 0\}$.

cylindrical coord.

$I = \iiint\limits_B z\sqrt{x^2+y^2}dxdydz = \iiint\limits_{B^*} zr^2 dr d\theta dz$ where$B^* = \{2r^2+z^2 \leq 3, z \geq r \geq 0, \theta \in [0,\pi]\}$

intersection of the line and ellipse $z = r = 1; 0 \leq r \leq 1; r \leq z \leq \sqrt{3-2r^2}$

so $I = \pi \int\limits_0^1 r^2 \left(\int\limits_r^{\sqrt{3-2r^2}} zdz\right) dr = \tfrac{\pi}{2} \int\limits_0^1 r^2\left(3-2r^2-r^2\right) dr = \tfrac{3\pi}{2} \int\limits_0^1 \left(r^2-r^4\right) dr =$

$= \tfrac{3\pi}{2}\left[\tfrac{1}{3}-\tfrac{1}{5}\right] = \dfrac{\pi}{5}.$