THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
FINAL EXAMINATION
MATH 353 (L60)

Summer 99 **TIME: 3 hours**
Each question – 10 marks.

1. Write the integral $I = \iint\limits_D \sqrt{x^2+y^2}\,dxdy$,

   where D is the region above the line $y=-x$,below the x-axis and inside the circle with the radius 2 and the center at $(2,0)$

   (a) as iterated integrals in Cartesian coordinates;
   (b) write the integral I in polar coordinates;
   (c) evaluate I.

Solution

Sketch the region $y \geq -x, y \leq 0$ and $(x-2)^2 + y^2 \leq 4$

find the intersection of the line and the circle

$(x-2)^2 + x^2 = 4 \qquad 2x(x-2) = 0 \qquad$ at $(2,-2)$ and $(0,0)$

we can see that for a) it is easier to slice it

horizontally

$D = \left\{-2 \leq y \leq 0, -y \leq x \leq 2 + \sqrt{4-y^2}\right\}$

and $I = \int\limits_{-2}^{0} \left(\int\limits_{-y}^{2+\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx \right) dy$

For b)

$-x \leq y \leq 0 \to\to\to\to -\frac{\pi}{4} \leq \theta \leq 0$

$(x-2)^2 + y^2 \leq 4 \to\to x^2 + y^2 \leq 4x \to\to\to r^2 \leq 4r\cos\theta$

$D^* = \left\{-\frac{\pi}{4} \leq \theta \leq 0, 0 \leq r \leq 4\cos\theta\right\}$ and

$I = \int\limits_{-\frac{\pi}{4}}^{0} \left(\int\limits_{0}^{4\cos\theta} r^2 dr \right) d\theta =$ (and for c)) $= \int\limits_{-\frac{\pi}{4}}^{0} \frac{4^3}{3}\cos^3\theta\; d\theta =$

$= \frac{64}{3} \int\limits_{-\frac{\pi}{4}}^{0} \left(1-\sin^2\theta\right)\cos\theta\; d\theta = \frac{64}{3} \int\limits_{-\frac{1}{\sqrt{2}}}^{0} (1-u^2)\,du = \frac{64}{3}\left[u - \frac{u^3}{3}\right]_{-\frac{1}{\sqrt{2}}}^{0} =$

$$= \frac{64}{3}\left[\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right] = \frac{64}{3\cdot 6\sqrt{2}}\left[6-1\right] = \frac{32\cdot 5}{9\sqrt{2}} = \frac{16\cdot 5\sqrt{2}}{9} = \frac{80\sqrt{2}}{9}.$$

2. Write the integral $\iiint\limits_B \sqrt{x^2+y^2}\, dxdydz$ where B is the ball $\{x^2+y^2+z^2 \le 4\}$

   (a) as iterated integrals in cylindrical coordinates ;

   (b) as iterated integrals in spherical coordinates;

   (c) evaluate I.

   Solution

   For a)

$$\iiint\limits_B \sqrt{x^2+y^2}\, dxdydz = \iiint\limits_{B^*} r^2 dr d\theta dz = \int_0^{2\pi} d\theta \cdot \int_0^2 r^2 \left(\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \right) dr$$

   where $B^* = \{0 \le \theta \le 2\pi, 0 \le r^2 + z^2 \le 4\}$

   For b)

$$\iiint\limits_B \sqrt{x^2+y^2}\, dxdydz = \iiint\limits_{B^{**}} \rho^3 \sin^2\phi \; d\rho d\phi d\theta \text{ since } x^2+y^2 = \rho^2 \sin^2\phi$$

   where $B^{**} = \{0 \le \theta \le 2\pi, 0 \le \rho \le 2, 0 \le \phi \le \pi\}$

   and for c)

$$\iiint\limits_{B^{**}} \rho^3 \sin^2\phi \; d\rho d\phi d\theta = 2\pi \int_0^2 \rho^3 d\rho \cdot \int_0^{\pi} \frac{1-\cos 2\phi}{2} d\phi = 8\pi\left[\frac{\pi}{2} - 0\right] = 4\pi^2.$$

3. For $\mathbf{F}(x,y) = (2y - x^2 y, 2x - y^3)$ and c given as the boundary of $D = \{-2 \le x \le 2, x^2 \le y \le 4\}$ oriented counterclockwise

   evaluate $\int\limits_c \mathbf{F}\cdot \mathbf{ds}$.

   Solution

   Since the curve is closed in the xy-plane,oriented counterclockwise

   we can use Green's theorem and

$$\int\limits_c \mathbf{F}\cdot\mathbf{ds} = \iint\limits_D \left[(F_2)_x - (F_1)_y\right] dxdy = \iint\limits_D \left[2 - 2 + x^2\right] dxdy$$

   where $D = \{-2 \le x \le 2, x^2 \le y \le 4\}$ –the region inside the curve

   so

$$\int_c \mathbf{F}\cdot\mathbf{ds} = \int_{-2}^{2} x^2 \left(\int_{x^2}^{4} dy\right) dx = 2\int_0^2 (4x^2 - x^4)\, dx = \tfrac{8}{3}\cdot 2^3 - \tfrac{2}{5}\cdot 2^5 =$$

$$= 2^6\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2^7}{15} = \frac{128}{15}.$$

(a) Describe the curve c given by $\mathbf{r}(t) = (t^2, t, 3t-1)$ in Cartesian coordinates x,y,z;

(b) calculate the arclength of c between $A\,(0,0,-1)$ and $B\,(1,-1,-4)$.

Solution

for a)

$x = t^2, y = t, z = 3t - 1$ gives $y = x^2$ and $z = 3y - 1$

so the curve is the intersection of a vertical paraboloid sheet and a plane.

for b)

$\mathbf{r}(t) = (t^2, t, 3t-1)$ and $\mathbf{r}'(t) = (2t, 1, 3)$, then

$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 10}$, $t = 0$ for $A, t = -1$ for B

and the length $l = \displaystyle\int_{-1}^{0} \sqrt{4t^2+10}\; dt = (u = 2t) = \tfrac{1}{2}\int_{-2}^{0} \sqrt{u^2+10}du =$

(Table) $= \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2+10} + 5\ln\left(u + \sqrt{u^2+10}\right)\right]_{-2}^{0} = \frac{1}{2}\sqrt{14} + \frac{5}{2}\ln\frac{\sqrt{10}}{\sqrt{14}-2}.$

4. For a conservative field $\mathbf{F}\,(x,y,z) = (y+z, x+e^y, x+\cos z)$ find the potential f such that $f\,(0,0,0) = 0$.

   Solution

   we know that $f_x = y + z$ so $f = xy + xz + c(y,z)$

   also $f_y = x + 0 + \frac{\partial c}{\partial y} = x + e^y$ so $\quad f = xy + xz + e^y + c(z)$

   finally $f_z = x + c'\,(z) = x + \cos z$ thus $f(x,y,z) = xy + xz + e^y + \sin z + c$

   now $f\,(0,0,0) = 1 + c$ therefore it must $c = -1$ and together

   $f(x,y,z) = xy + xz + e^y + \sin z - 1.$

5. Show that for any smooth (i.e. with continuous second order partials) vector field $\mathbf{F}$ of 3 variables and any smooth real-valued function $h(x,y,z)$

   $div(h\ \mathbf{F}) = gradh\ \bullet\mathbf{F} +\ h(div\mathbf{F})$ i.e. $\quad \nabla\bullet(h\ \mathbf{F}) = \nabla h\ \bullet\mathbf{F} + h(\nabla\bullet\mathbf{F})$.

   Solution

   for $h\mathbf{F}\,(x,y,z) = (hF_1, hF_2, hF_3)$ ..all functions of x, y, z

   L.H.S $= div(h\ \mathbf{F}) = (hF_1)_x + (hF_2)_y + (hF_3)_z =$

   $= h_xF_1 + h\,(F_1)_x + h_yF_2 + h\,(F_2)_y + h_zF_3 + h\,(F_3)_z =$

   $= (h_x, h_y, h_z)\bullet(F_1, F_2, F_3) + h\left[(F_1)_x + (F_2)_y + (F_3)_z\right]$ =R.H.S.

6. Evaluate $\oint_c \mathbf{F} \bullet \mathbf{ds}$ where $\mathbf{F} = (x, y + e^y, xz)$ and the curve c is the intersection of the plane $2y + z = 3$ and the paraboloid $z = x^2 + y^2$, oriented clockwise.

Solution

the curve is closed so we can use Stokes' theorem

we need $curl\ \mathbf{F} = \begin{Vmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x & y + e^y & xz \end{Vmatrix} = (0, -z, 0)$

and we must define S (notice that we have 2 possibilities–plane or paraboloid)

S is a part of the plane $z = 3 - 2y$ inside the paraboloid $z \geq x^2 + y^2$

we have to specify $D : 3 - 2y \geq x^2 + y^2 \quad 4 \geq x^2 + (y + 1)^2$

and $\mathbf{n} = (\nabla z, -1) = (0, -2, -1)$ since clockwise orientation

$$\oint_c \mathbf{F} \cdot \mathbf{ds} = \iint_S curl\ \mathbf{F} \bullet \mathbf{dS} = \iint_D 2z\ dxdy = (\text{ subst for } z) =$$

$$= \iint_D 2(3 - 2y)\, dxdy = 2 \iint_D [5 - 2(y + 1)]\, dxdy = (Y = y + 1)$$

$$= 10 \cdot \text{area of } D - 4 \int\int_{\{x^2+Y^2 \leq 4\}} Y dxdY = 40\pi - 4 \cdot 0 = 40\pi.$$

7. Calculate the surface area of the lateral part of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 9$, in the first octant.

Solution

for the surface $S : z = x^2 + y^2$ we need $\mathbf{n} = (\nabla z, -1) = (2x, 2y, -1)$

and D

we know that $1 \leq z \leq 9$ so $D = \{1 \leq x^2 + y^2 \leq 9, x > 0, y > 0\}$

and $SA = \iint_S dS = \iint_D \|\mathbf{n}\|\, dxdy = \iint_D \sqrt{4x^2 + 4y^2 + 1} dxdy =$

(polar coord)$= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_1^3 r\sqrt{4r^2 + 1} dr = \frac{\pi}{2} \left[\frac{(4r^2 + 1)^{\frac{3}{2}}}{3 \cdot 4} \right]_1^3 =$

$= \frac{\pi}{24} \left[37\sqrt{37} - 5\sqrt{5} \right].$

8. Find the flux $\iint_S \mathbf{F} \cdot \mathbf{dS}$ of $\mathbf{F}(x, y, z) = (xz, yz, 3z)$ outward where the surface $S = \{x^2 + y^2 + z^2 = 4, z \leq 0\}$.

Solution

to describe S $\quad z=-\sqrt{4-x^2-y^2},(x,y)\in\{x^2+y^2\leq 4\}$

$\mathbf{n}=(\nabla z,-1)=\left(\frac{x}{\sqrt{4-x^2-y^2}},\frac{y}{\sqrt{4-x^2-y^2}},-1\right)$ since outward means downward

and on S $\quad \mathbf{F}=\left(-x\sqrt{4-x^2-y^2},-y\sqrt{4-x^2-y^2},-3\sqrt{4-x^2-y^2}\right)$

$\mathbf{F}\bullet\mathbf{n}=-x^2-y^2+3\sqrt{4-x^2-y^2}$

finally

$$\iint_S \mathbf{F}\bullet d\mathbf{S}=\iint_D\left(-x^2-y^2+3\sqrt{4-x^2-y^2}\right)dxdy=(\text{ polar coord.})$$

$$=2\pi\int_0^2\left(-r^2+3\sqrt{4-r^2}\right)rdr=2\pi\left[-\frac{r^4}{4}-(4-r^2)^{\frac{3}{2}}\right]_0^2=-8\pi+16\pi=8\pi.$$

9. Calculate the total flux of $\mathbf{F}(x,y,z)=\left(yz,\sqrt{y},3x\right)$ coming out from the closed cylinder (incl.top and bottom) $\{x^2+z^2=9,0\leq y\leq 4\}$.

Solution

Since the surface is closed we can use the divergence (Gauss') theorem and

$$flux=\iint_S \mathbf{F}\bullet d\mathbf{S}=\iiint_B div\mathbf{F}\ dxdydz \text{ where } B \text{ is the inside the cylinder}$$

$B=\{x^2+z^2\leq 9,0\leq y\leq 4.\}$ and $div\mathbf{F}=0+\frac{1}{2\sqrt{y}}+0$

therefore

$$flux=\int\int_{\{x^2+z^2\leq 9\}}dxdz\cdot\int_0^4\frac{1}{2\sqrt{y}}dy=9\pi\cdot\left[\sqrt{y}\right]_0^4=18\pi.$$