MATH 353
Handout #2 Solutions

1. Find absolute extrema of $f(x,y) = \frac{1}{8}x^3 + y^3$ on the circle (disk) $x^2 + y^2 \leq 65$

   SOLUTION:

   The function is continuous and the set is closed and bounded. The first C.P. is on the inside. Solve $\nabla f = \mathbf{0}$, so

   $$\frac{3}{8}x^2 = 0 \qquad 3y^2 = 0$$

   The CP is $(0,0)$

   For C.P on the boundary use Langrange multiplier method where $g(x,y) = x^2 + y^2 = 65$. Solve $\nabla f = \lambda \nabla g$

   $\frac{3}{8}x^2 = \lambda 2x \longrightarrow\longrightarrow\longrightarrow x = 0$ or $\frac{3}{8}x = 2\lambda$

   $3y^2 = \lambda 2y \longrightarrow\longrightarrow\longrightarrow y = 0$ or $3y = 2\lambda$

   If $x = 0$ back to the circle; $y = \pm\sqrt{65}$. Similarly for $y = 0$, then $x = \pm\sqrt{65}$.

   For $xy \neq 0$ then $2\lambda = \frac{3}{8}x = 3y \Longrightarrow x = 8y$ back to the circle 64 $y^2 + y^2 = 65$. Thus $y = \pm 1$ and $x = \pm 8$.

   There are 7 critical points:$(0,0), \left(0, \pm\sqrt{65}\right), \left(\pm\sqrt{65}, 0\right), (\pm 8, \pm 1)$.

   Test values of f : ..0...... $\pm 65\sqrt{65}$.... $\pm \frac{65}{8}\sqrt{65}$..... ± 65

   So **max** is $65\sqrt{65}$ at $\left(0, \sqrt{65}\right)$ and **min** is $-65\sqrt{65}$ at $\left(0, -\sqrt{65}\right)$.

2. Find the absolute extrema of $f(x,y) = x^2 + y^2$

   on the surface $S = \left\{\frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0.\right\}$.

   SOLUTION

   Notice that we need $x, y \geq 0$ to make S bounded. So $g(x,y) = \frac{1}{8}x^3 + y^3 = 65, x \geq 0, y \geq 0$

   Solve $\nabla f = \lambda \nabla g$

   $2x = \lambda\frac{3}{8}x^2. \longrightarrow\longrightarrow\longrightarrow x = 0$ or $\frac{16}{x} = 3\lambda$

   $2y = \lambda 3y^2 \longrightarrow\longrightarrow\longrightarrow\longrightarrow y = 0$ or $\frac{2}{y} = 3\lambda$

For $x = 0$ back to S $\quad y = \sqrt[3]{65} = 4.02$. Similarly for $y = 0, x = 2\sqrt[3]{65}$

For $xy \neq 0$ we have

$3\lambda = \frac{16}{x} = \frac{2}{y}$ $\quad$ so $x = 8y$ back to S $\quad 64y^3 + y^3 = 65$

$y = 1, x = 8$

3 critical points :$\left(0, \sqrt[3]{65}\right)$ $\quad \left(2\sqrt[3]{65}, 0\right)$ $\quad (8, 1)$

values of f $\quad 65^{\frac{2}{3}} = 16.1$ $\quad 4 \cdot 65^{\frac{2}{3}} = 64.66$ $\quad 65$

So **max** is 65 at $(8, 1)$ and **min** is $65^{\frac{2}{3}}$ at $\left(0, \sqrt[3]{65}\right)$.

3. Find absolute maxim and minima of $f(x, y) = 2y^2 - x + x^2$

   inside and on the triangle T with vertices $O(0, 0)$, $A(1, 1)$, $B(1, -1)$.

   SOLUTION

   Since the function is continuous and the set is closed and bounded we have to find all critical points inside and on the boundary,and check the values at those.

   For critical points inside solve $\nabla f = \mathbf{0}$

   $f_x = 2x - 1 = 0, f_y = 4y = 0$ so $x = \frac{1}{2}, y = 0$

   Now, the boundary of T consists of 3 line segments

   $B_1 = \{y = x, 0 \leq x \leq 1\}$ and $B_2 = \{y = -x, 0 \leq x \leq 1\}$

   $f(x, \pm x) = 3x^2 - x = h(x)$,$h'(x) = 6x - 1 = 0$ for $x = \frac{1}{6} \rightarrow y = \pm\frac{1}{6}$

   and the ends(corners)

   $B_3 = \{x = 1, -1 \leq y \leq 1\}$ and f on B_3 is $f(1, y) = g(y) = 2y^2$

   and $g'(x) = 4y = 0$ for $y = 0$, $x = 1$

   Together all critical points in T and on ∂T are

   $\left(\frac{1}{2}, 0\right), \left(\frac{1}{6}, \pm\frac{1}{6}\right), (0, 0), (1, \pm 1), (1, 0)$

   Check the values of f

   ......$\frac{-1}{4}$.....$\frac{-1}{12}$ ....0..........2........0

   So abs.max.value is 2 at the points $(1, \pm 1)$

   and abs.min. value is $\frac{-1}{4}$ at the point $\left(\frac{1}{2}, 0\right)$.

4. Find the point on the plane $x - 2y - z = 3$ closest to the point $P\,(1, -1, 2)$.

   Justify!

   Solution:

   We are looking for minimum of the distance to P or the square of distance

   $f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-2)^2$ and the constraint $g(x, y, z) = x - 2y - z = 3$

   Solve $\nabla f = \lambda \nabla g$

   $2\,(x-1) = \lambda$

   $2\,(y+1) = -2\lambda$.................$\lambda = 2x - 2 = -y - 1 = 4 - 2z$

   $2(z-2) = -\lambda$..........................$1 - 2x = y$.........$z = 3 - x$

   back to the plane

   $x - 2(1-2x) - (3-x) = 6x - 5 = 3 \to\to\to x = \frac{4}{3}, y = \frac{-5}{3}, z = \frac{5}{3}$

   and the C.P. point is $\left(\frac{4}{3}, \frac{-5}{3}, \frac{5}{3}\right)$

   To justify that we have found minimum

   the set is not bounded but if we take a bounded part we have to have maximum and minimum

   but the maximum will be on the boundary since the distance is increasing when we move far away

   therefore the critical point must be minimum.

5. Find absolute maximum of $f(x, y, z) = xyz$

   largest box with sides x, y, z on $\{(x, y, z)\,; 2xy + 2xz + 3yz = 144, x \geq 0, y \geq 0, z \geq 0\}$

   (You may assume that there is an absolute maximum)

   Solution:

   $\nabla f = \lambda \nabla g$ where $g(x, y, z) = 2xy + 2xz + 3yz = 144$

   $yz = \lambda(2y + 2z)$ if $y = 0$ or $z = 0$ or $x = 0$ then $f = 0$

   so we can assume that all $xyz \neq 0$

$xz = \lambda(2x + 3z)$ $\qquad \frac{1}{\lambda} = \frac{2y+2z}{yz} = \frac{2x+3z}{xz} = \frac{2x+3y}{xy}$

$xy = \lambda(2x + 3y)$

from the first equality (cancel z) ......$2yx+2xz = 2xy+3zy \to 2x = 3y$

from the second equality(cancel x)....$2xy + 3zy = 2xz + 3zy \to z = y$

back to the surface $3z^2 + 3z^2 + 3z^2 = 9z^2 = 144 \to z^2 = \frac{144}{9} = 16 \to z = \pm 4$

and critical points are $(\pm 6, \pm 4, \pm 4)$,values of f are ± 96

so maximum is 96 at the point $(6, 4, 4)$.

6. (a) Evaluate $\int_1^3 \left(\int_{-x}^{x^2} xe^{2y} dy \right) dx$.

   (b) Switch the order of integration in the integral above and sketch the region D.

   Solution for (a):

$I = \int_1^3 \left(\int_{-x}^{x^2} xe^{2y} dy \right) dx = \int_1^3 x \left(\int_{-x}^{x^2} e^{2y} dy \right) dx. = \int_1^3 x \left(\left[\frac{e^{2y}}{2} \right]_{y=-x}^{y=x^2} \right) dx =$
$\frac{1}{2} \int_1^3 x \left(e^{2x^2} - e^{-2x} \right) dx =$

$= \frac{1}{2} \int_1^3 xe^{2x^2} dx \,(\text{subst}) - \frac{1}{2} \int_1^3 xe^{-2x} dx \,(\text{byparts}) =$

$= \frac{1}{8} \left[e^{2x^2} \right]_1^3 - \frac{1}{2} \left[x \frac{e^{-2x}}{-2} \right]_1^3 - \frac{1}{4} \int_1^3 e^{-2x} dx = \frac{1}{8} \left[e^{18} - e^2 \right] + \frac{1}{4} \left[3e^{-6} - e^{-2} \right] +$
$\frac{1}{8} \left[e^{-6} - e^{-2} \right] = ..$

For (b):

since we have three "left ends" we have to split the domain $D = D_1 \cup D_2$ $\cup D_3$where

$D_1 = \{(x, y) \,; -3 \le y \le -1, -y \le x \le 3\}$, $D_2 = \{(x, y) \,; -1 < y \le 1, 1 \le x \le 3\}$

and $D_3 = \left\{(x, y) \,; 1 < y \le 9, \sqrt{y} \le x \le 3\right\}$

and the integral

$I = \int_{-3}^{-1} \left(\int_{-y}^3 xe^{2y} dx \right) dy + \int_{-1}^1 \left(\int_1^3 xe^{2y} dx \right) dy + \int_1^9 \left(\int_{\sqrt{y}}^3 xe^{2y} dx \right) dy.$

7. Evaluate $\int \int_D \sqrt{2 - x^2} dA$ where D is smaller region between $y = x^2$ and $x^2 + y^2 = 2$.

and sketch the region

Solution:

find the intersection of the parabola $y = x^2$ and the circle $x^2 + y^2 = 2$

$y^2 + y - 2 = 0 \qquad (y-1)(y+2) = 0$

but y must be positive so $y = 1$ and $x = \pm 1$ and $D = \{-1 \le x \le 1$
$x^2 \le y \le \sqrt{2 - x^2}\}$

$\int\int_D \sqrt{2-x^2} dA = \int_{-1}^{1} \sqrt{2-x^2} \left(\int_{x^2}^{\sqrt{2-x^2}} dy \right) dx = \int_{-1}^{1} \sqrt{2-x^2} \left(\sqrt{2-x^2} - x^2 \right) dx =$

$= \int_{-1}^{1} (2 - x^2 - x^2 \sqrt{2-x^2}) dx = (\text{evenf.}) = 2 \int_0^1 ...dx =$

$= 2 \cdot 2 - 2 \left[\frac{x^3}{3} \right]_0^1 - 2 \left(\text{Table} a = \sqrt{2} \right) \left[\frac{x}{8} (2x^2 - 2) \sqrt{2-x^2} + \frac{1}{2} \arcsin \frac{x}{\sqrt{2}} \right]_0^1 =$

$= 4 - \frac{2}{3} + 0 - \arcsin \frac{1}{\sqrt{2}} = \frac{10}{3} - \frac{\pi}{4}.$

8. Switch the order of integration in the integral $\int_0^{\frac{\pi}{4}} \left(\int_0^{\tan x} f(x,y) dy \right) dx$.

   Solution:

   given $\qquad 0 \le x \le \frac{\pi}{4}$ and $\ 0 \le y \le \tan x$ sketch

   $y = \tan x$ is equivalent to $\arctan y = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ and $\tan \frac{\pi}{4} = 1$

   so $\qquad 0 \le y \le 1$ and $\qquad \arctan y \le x \le \frac{\pi}{4}$

   and the integral

   $\int_0^{\frac{\pi}{4}} \left(\int_0^{\tan x} f(x,y) dy \right) dx = \int_0^1 \left(\int_{\arctan y}^{\frac{\pi}{4}} f(x,y) dx \right) dy.$

9. For $\int\int_D \frac{1}{x^2+y}\ dA$ where D is the region between the x-axis and $y = 4 - x^2$

   sketch the region D and set up BOTH iterated integrals and evaluate one of them.

   (Hint: $\lim_{x \to 0^+} x \ln x = 0$).

   Solution:

   the region is above x-axis and below parabola $y = 4 - x^2$

   so $\qquad -2 \le x \le 2 \qquad 0 \le y \le 4 - x^2$ or $0 \le y \le 4 \qquad -\sqrt{4-y} \le x \le \sqrt{4-y}$

and

$\int\int_D \frac{1}{x^2+y}\ dA = \int_{-2}^{2}\left(\int_0^{4-x^2} \frac{1}{x^2+y}dy\right)dx = \int_0^4\left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2+y}dx\right)dy$

evaluate the first ordered iterated integrals

$\int_{-2}^{2}\left(\int_0^{4-x^2} \frac{1}{x^2+y}dy\right)dx = \int_{-2}^{2}\left[\ln\left(x^2+y\right)\right]_{y=0}^{y=4-x^2}dx = \int_{-2}^{2}\left(\ln 4 - \ln x^2\right)dx =$
$4\ln 4 - 2\int_0^2 \ln x^2 dx =$

$= 4\ln 4 - 4\int_0^2 \ln x dx = 8\ln 2 - 4\left[x\ln x - x\right]_0^2 = 8.$ (Otherwise $\ln x^2 =$ $2\ln|x|!$),

using the limit of $x\ln x \to 0$ as $x \to 0^+$.

The other way

$\int_0^4\left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{x^2+y}dx\right)dy = \int_0^4\left(\left[\frac{1}{\sqrt{y}}\arctan\frac{x}{\sqrt{y}}\right]_{x=-\sqrt{4-y}}^{x=\sqrt{4-y}}\right)dy =$ ....harder

10. Calculate the volume of the solid below the surface $z = e^{(y-1)^2}$ and above

the triangle T with vertices $A(-1,0), B(0,1), C(2,0)$ with vertical sides.

Solution:

$V = \int\int_T e^{(y-1)^2}\ dxdy$ it is easier to slice the triangle horizontally

$0 \le y \le 1$ $\quad line_{AB} \le x \le line_{BC}$

where line AB $\quad y = x+1$ or $x = y-1$

lineBC $\quad y = 1 - \frac{1}{2}x$ or $x = 2-2y$

so $\quad V = \int_0^1\left(e^{(y-1)^2}\int_{y-1}^{2-2y} dx\right)dy = \int_0^1 e^{(y-1)^2}(3-3y)\,dy = -\frac{3}{2}\int_1^0 e^u du =$
$\frac{3}{2}[e-1]$

by subst. $u = (y-1)^2$