The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #3 Solution

1. The intersection of the two surfaces happens when $z = \cos\sqrt{x^2+y^2} = 0$. Thus $\sqrt{x^2+y^2} = \frac{\pi}{2} + k\pi$,we can choose $k = 0$. So $D = \{(x,y); x^2 + y^2 \le \left(\frac{\pi}{2}\right)^2\}$ and

$$V = \iint_D \cos\sqrt{x^2+y^2}dxdy = \iint_{D^*} r\cos r \ drd\theta$$

(polar coord.) where $D^* = \{(r,\theta); 0 < r \le \frac{\pi}{2}; 0 \le \theta < 2\pi\}$. By parts $V = 2\pi\int_0^{\frac{\pi}{2}} r\cos r \ dr = 2\pi\left[r\sin r - \iint \sin r \ de\right]_0^{\frac{\pi}{2}} = \pi^2 + 2\pi\left[\cos r\right]_0^{\frac{\pi}{2}} = \pi^2 - 2\pi$.

2. Using polar coord.

$$\iint_D e^{3(x^2+y^2)}dxdy = \iint_{D^*} e^{3r^2} r \ drd\theta = \pi\left[\frac{e^{3r^2}}{6}\right]_1^2 = \frac{\pi}{6}\left(e^{12} - e^3\right)$$

where $D^* = \{0 \le \theta \le \pi, 1 \le r \le 2\}$

3. The triangle is $T = \{0 \le x \le 2, 2x \le y \le 4\}$ so

$$I = \iint_T \frac{1}{(y-2x)^k} \ dA = \int_0^2 (\int_{2x}^4 (y-2x)^{-k} \ dy)dx = \int_0^2 \left[\frac{(y-2x)^{1-k}}{1-k}\right]_{y\to 2x}^{y=4} dx$$

for $k \ne 1$.

Then $\lim_{y\to 2x}(y-2x)^{1-k} = 0$ for $1-k > 0$ and diverges for $1-k < 0$.

For $k = 1$, the anti-derivative is a logarithm and $[\ln(y-2x)]_{y\to 2x}^{y=4} = +\infty$. So the integral is convergent for only $k < 1$ and

$$I = \int_0^2 \left[\frac{(4-2x)^{1-k}}{1-k}\right] dx = \left[\frac{(4-2x)^{2-k}}{(-2)(1-k)(2-k)}\right]_0^2 = \frac{2}{(1-k)(2-k)}$$

since $2-k > 0$.

4. Sketch the set. The intersection of the line $x = 1$ and the circle is at the point $(1, 1)$. Use polar coord. with $D^* = \{0 \leq r \leq \sqrt{2}, \theta \in \left[0, \frac{\pi}{4}\right], r \geq \frac{1}{\cos\theta}\}$

   The necessary cond.from $\frac{1}{\cos\theta} \leq r \leq \sqrt{2}$ is $\cos\theta \geq \frac{1}{\sqrt{2}}$ so $\theta \in \left[0, \frac{\pi}{4}\right]$

   $\iint_D \frac{dxdy}{\sqrt{x^2+y^2}} = \iint_{D^*} \frac{rdrd\theta}{\sqrt{r^2}} = \iint_0^{\frac{\pi}{4}} \left(\iint_{\frac{1}{\cos\theta}}^{\sqrt{2}} dr\right) d\theta = \iint_0^{\frac{\pi}{4}} \left(\sqrt{2} - \frac{1}{\cos\theta}\right) d\theta =$

   (Table) $= \frac{\pi\sqrt{2}}{4} - [\ln|\sec\theta + \tan\theta|]_0^{\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4} - \left[\ln\left|\sqrt{2}+1\right| - \ln 1\right] =$

   $= \frac{\pi\sqrt{2}}{4} - \ln\left(\sqrt{2}+1\right)$

5. Using polar coord. $\iint_D \frac{x\sin\pi\left(x^2+y^2\right)}{\sqrt{x^2+y^2}} dxdy = \iint_{D^*} \frac{r\cos\theta\sin\left(\pi r^2\right)}{r} rdrd\theta ==$

   $\iint_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta d\theta \cdot \iint_0^1 r\sin\pi r^2 \; dr = [\sin\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \left[\frac{-\cos\pi r^2}{2\pi}\right]_0^1 = \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\pi}\right).$

   where $D^* = \{\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, 0 < r \leq 1\}$

6. $D = \{x \in (1, +\infty), 0 \leq y \leq \frac{1}{x^2}\}$ is unbounded so

   $I = \iint_D e^{-x^2y} dA = \iint_1^\infty (\iint_0^{\frac{1}{x^2}} e^{-x^2y} dy) dx = \iint_1^\infty \left[\frac{e^{-x^2y}}{-x^2}\right]_{y=0}^{y=\frac{1}{x^2}} dx =$

   $\iint_1^\infty \left[\frac{e^{-1}-1}{-x^2}\right] dx == \left(\frac{1}{e} - 1\right)\left[\frac{1}{x}\right]_{x=1}^{x\to\infty} = 1 - \frac{1}{e}.$

7. (a) Sketch the set; the intersection of the circle and the line $y = 1$ is at $x = \pm 1$ so $\iint_D (x^2+y^2) dxdy = \iint_{-1}^1 \left(\iint_1^{\sqrt{2-x^2}} \left(x^2+y^2\right) dy\right) dx =$

   $\iint_{-1}^1 \left(x^2\sqrt{2-x^2} + \frac{1}{3}\left(2-x^2\right)^{\frac{3}{2}} - x^2 - \frac{1}{3}\right) dydx = ...$

   (b) In polar coord. $\iint_D (x^2+y^2) dxdy = \iint_{D^*} r^3 drd\theta$

   where $D^* = \{0 < r \leq \sqrt{2}\ ,\ r \geq \frac{1}{\sin\theta}, \theta \in \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]\}$. Since $y \geq 1$ implies $r\sin\theta \geq 1$, $\sin\theta$ must be positive and $\sqrt{2} \geq \frac{1}{\sin\theta} =$

   $\iint_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\iint_{\frac{1}{\sin\theta}}^{\sqrt{2}} r^3 \; dr) d\theta = \frac{1}{4}\iint_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 - \csc^4\theta) d\theta = \frac{\pi}{2} - \frac{2}{4}\iint_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1+\cot^2\theta\right)\csc^2\theta d\theta =$

   $1 + \cot^2\theta = \csc^2\theta$, symmetry. Subst. $u = \cot\theta$, $du = -\csc^2\theta d\theta$

   $= \frac{\pi}{2} - \frac{1}{2}\iint_0^1 \left(1+u^2\right) du = \frac{\pi}{2} - \frac{2}{3}.$

8. $D = \{x \in (1, +\infty), 0 \leq y \leq x\}$ is unbounded.

   $I = \iint_D e^{-x-y} dA = \iint_1^\infty e^{-x} (\iint_0^x e^{-y} dy) dx = \iint_1^\infty e^{-x} \left[-e^{-y}\right]_{y=0}^{y=x} dx =$

   $\iint_1^\infty \left[e^{-x} - e^{-2x}\right] dx == \left[-e^{-x} + \frac{1}{2}e^{-2x}\right]_{x=1}^{x\to\infty} = \frac{1}{e} - \frac{1}{2e^2} \qquad (e^{-\infty} = 0)$

9. (a) Sketch the set.

   So $\iint_D \frac{1}{(x^2+y^2)^2} dxdy = \iint_0^1 \left(\iint_{1-x}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2)^2} dy \right) dx =$ difficult!

   (b) In polar coord. $\iint_D \frac{1}{(x^2+y^2)^2} dxdy = \iint_{D^*} r^{-3} dr d\theta$

   where $D^* = \{0 < r \leq 1 \,,\, r \geq \frac{1}{\cos\theta + \sin\theta}, \theta \in \left[0, \frac{\pi}{2}\right]\} = \iint_0^{\frac{\pi}{2}} (\iint_{\frac{1}{\cos\theta+\sin\theta}}^1 r^{-3}$ $dr) d\theta = -\frac{1}{2} \iint_0^{\frac{\pi}{2}} (1-(\cos\theta + \sin\theta)^2) d\theta = \frac{1}{2} \iint_0^{\frac{\pi}{2}} \sin 2\theta d\theta = \frac{1}{4} \left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} =$ $\frac{1}{2}$.

10. The function is unbounded

    $I = \iint_D \frac{1+\ln x}{y} dA = \iint_0^1 \frac{1}{y} (\iint_0^{e^y} (1 + \ln x)\, dx) dy = \iint_0^1 \frac{1}{y} \left[x \ln x\right]_0^{e^y} dy =$ $\iint_0^1 \frac{1}{y} \left(e^y y\right) dy$

    $= e - 1 \qquad (\lim_{x \to 0+} x \ln x = 0)$