The University of Calgary
Department of Mathematics and Statistics
MATH 353 Handout #4 Solution

1. Given $\mathbf{F}(x, y, z) = (3x^2yz, kyz + x^3z, x^3y + 1 + y^2)$.

   (a) Find the value of k so that the field $\mathbf{F}$ is conservative.

   (b) Then, find a potential of $\mathbf{F}$.

   For 1a)

   necessary condition $(F_1)_y = 3x^2z = (F_2)_x = 3x^2z$

   $(F_1)_z = 3x^2y = (F_3)_x = 3x^2y$, and finally

   $(F_2)_z = ky + x^3 = (F_3)_y = x^3 + 2y$ gives us $k = 2$.

   For 1b)

   $f_x = F_1 = 3x^2yz$ so by integrating with respect to x :

   $f(x, y, z) = \int(3x^2yz\ dx + c(y, z) = x^3yz + c(y, z)$

   differentiate $f_y = F_2 = 2yz + x^3z = x^3z + \frac{\partial c}{\partial y}$ thus $\frac{\partial c}{\partial y} = 2yz$ and $c(y, z) = y^2z + c(z)$

   together $f(x, y, z) = x^3yz + y^2z + c(z)$

   differentiate $f_z = F_3 = x^3y + 1 + y^2 = x^3y + y^2 + c'(z)$ thus $c' = 1$ and $c(z) = z + c$

   and finally the general potential $f(x, y, z) = x^3yz + y^2z + z + c$ where c is any constant.

2. Evaluate $\int_c f\ ds$ where $f(x, y, z) = z^2$ and c is the part of the line of intersection of two planes;

   $x + y - z = 1$ and $2x + y - 3z = 0$ between the xy-plane and the point $D(3, 0, 2)$

   For 2)

   first let's find the line ,a direction vector $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$ (of normal vectors of the given planes)

$\mathbf{d} = (1, 1, -1) \times (2, 1, -3) = (-2, 1, -1)$ so

$(x, y, z) = (3, 0, 2) + t\,(-2, 1, -1)$ and when $t = 0$ we get D

$x = 3 - 2t, y = t, z = 2 - t$

to get a point on the xy-plane $z = 0$ so $t = 2$ and the point is $P\,(-1, 2, 0)$

Together $\mathbf{r}(t) = (3 - 2t, t, 2 - t) \quad t \in [0, 2]\,, \mathbf{r}'(t) = (-2, 1, -1) = \mathbf{d}$

the given function f evaluated on $c \quad f \circ \mathbf{r} = (2 - t)^2$

$$\int_c f\ ds = \int_0^2 (2 - t)^2\, \|\mathbf{r}'(t)\|\, dt = \sqrt{6}\left[\frac{(t - 2)^3}{3}\right]_0^2 = \frac{8}{3}\sqrt{6}.$$

3. For $\mathbf{F}(x, y) = (ky^2 + x, xy - \dfrac{1}{\sqrt{y}})$ find the value for k

   so that the field is conservative ,then find a potential.

   For 3)

   $F_1 = ky^2 + x$ and $F_2 = xy - \frac{1}{\sqrt{y}}$ for $y > 0$

   $(F_1)_y = 2ky = (F_2)_x = y$ so $k = \frac{1}{2}$

   then

   $$f_x = \frac{1}{2}y^2 + x \text{ and } f_y = xy - \tfrac{1}{\sqrt{y}}$$

   $f = \int f_x dx = \int \left(\frac{1}{2}y^2 + x\right) dx + c(y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 + c(y)$

   $f_y = xy + c'(y) = xy - \frac{1}{\sqrt{y}} \qquad c'(y) = -\frac{1}{\sqrt{y}}$ for $y > 0$

   and $c(y) = -2\sqrt{y} \qquad$ together $f(x, y) = \frac{1}{2}xy^2 + \frac{1}{2}x^2 - 2\sqrt{y} + const.$

4. Evaluate $\int_c z\ ds$ and c is the intersection of the plane $z - y = 1$ and the paraboloid $0 = x - y^2$ between $A(1, -1, 0)$ and $B\,(0, 0, 1)$.

   For 4)

   intersection of $\qquad z - y = 1$ and $0 = x - y^2$ between $A(1, -1, 0)$ and $B\,(0, 0, 1)$

   $y = t$ and $\mathbf{r}\,(t) = (t^2, t, 1 + t)$ for $t \in [-1, 0]$

   $\mathbf{r}'\,(t) = (2t, 1, 1)$ and $\|\mathbf{r}'(t)\| = \sqrt{2 + 4t^2}$

$\int_c z\ ds = \int\limits_{-1}^{0}(1+t)\sqrt{2+4t^2}dt = \int\limits_{-1}^{0}\sqrt{2+4t^2}dt + \int\limits_{-1}^{0} t\sqrt{2+4t^2}dt =$

$$\left(u = 2t, a = \sqrt{2}, dt = \frac{1}{2}du\text{OR } \sqrt{2+4t^2} = 2\sqrt{\frac{1}{2}+t^2}, a = \frac{1}{\sqrt{2}}\text{Table}\right)$$

$$= \frac{1}{2}\left[t\sqrt{2+4t^2} + \ln(2t+\sqrt{2+4t^2})\right]_{-1}^{0} + \left[\frac{(2+4t^2)^{\frac{3}{2}}}{12}\right]_{-1}^{0} =$$

$$= \frac{\sqrt{6}}{2} + \frac{1}{2}\ln\sqrt{2} - \frac{1}{2}\ln\left(\sqrt{6}-2\right) + \left[\frac{\sqrt{2}}{6} - \frac{\sqrt{6}}{2}\right] = \sqrt{\frac{3}{2}} - \frac{1}{2}\ln\left(\sqrt{3}-\sqrt{2}\right).$$

5. Find $\int_c \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{F}(x,y,z) = (z, e^{\frac{y}{x}}, 2x)$ is given by $\mathbf{r}(t) = (t, t^2, e^t), t \in [1,2]$.

For 5)

$\mathbf{r}(t) = (t, t^2, e^t), t \in [1,2] \qquad \mathbf{r}'(t) = (1, 2t, e^t)$

then the field on $c : \mathbf{F}\circ\mathbf{r} = (e^t, e^t, 2t)$

$\int_c \mathbf{F}\cdot d\mathbf{s} = \int_1^2 \mathbf{F}\cdot\mathbf{r}'dt = \int_1^2 (e^t + 4te^t)\,dt =$ (by parts)

$= [e^t + 4te^t - 4e^t]_1^2 = 5e^2 - e.$

6. For $\mathbf{F}(x,y) = (3x\sqrt{x^2+y^4} + \cos x, ky^3\sqrt{x^2+y^4} + \sin y)$ find the value for k

so that the field is conservative ,then find a potential.

For 6)

$F_1 = (3x\sqrt{x^2+y^4} + \cos x$and $F_2 = ky^3\sqrt{x^2+y^4} + \sin y$ for any point except the origin

$(F_1)_y = \dfrac{6xy^3}{\sqrt{x^2+y^4}} = (F_2)_x = \dfrac{kxy^3}{\sqrt{x^2+y^4}}$ so $k = 6$ then $f = ?$

$$f_x = (3x\sqrt{x^2+y^4} + \cos x \text{ and } f_y = ky^3\sqrt{x^2+y^4} + \sin y$$

$f = \int f_x dx = \int \left(3x\sqrt{x^2+y^4} + \cos x\right)dx + c(y) = (x^2+y^4)^{\frac{3}{2}} + \sin x + c(y)$

$f_y = \dfrac{3}{2}\sqrt{x^2+y^4}\cdot 4y^3 + c'(y) = F_2 \qquad c'(y) = \sin y$

and $\qquad c(y) = -\cos y$

together $f(x,y) = (x^2+y^4)^{\frac{3}{2}} + \sin x - \cos y + const.$

7. Evaluate $\int_c z\ ds$ and c is given by $\mathbf{r}(t) = (t\cos t, t\sin t, t)\,, t \in [0,1]\,.$

For 7)

$\mathbf{r}(t) = (t\cos t, t\sin t, t)\,, t \in [0,1]$ $\mathbf{r}'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$

$\|\mathbf{r}'(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} = \sqrt{2+t^2}$

$$\int_c z\ ds = \int_0^1 (t\sqrt{2+t^2}dt = \left[\frac{(2+t^2)^{\frac{3}{2}}}{3}\right]_0^1 = \sqrt{3} - \frac{2\sqrt{2}}{3}.$$

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8. Find $\int_c \mathbf{F}\cdot d\mathbf{s}$ where $\mathbf{F}(x,y,z) = (y, z, 2x - z)$ and c is the intersection of the plane $z = 2x$ and the paraboloid $z = x^2 + y^2$ oriented counter-clockwise.

for 8)

intersection of $z = 2x$ and $z = x^2 + y^2 \qquad 2x = x^2 + y^2$

$1 = (x-1)^2 + y^2 \qquad x = 1 + \cos t, y = \sin t$ and $z = 2x$

$\mathbf{r}(t) = (1 + \cos t, \sin t, 2 + 2\cos t)\,, t \in [0, 2\pi]$

$\mathbf{r}'(t) = (-\sin t, \cos t, -2\sin t)$

then the field on $c : \mathbf{F} \circ \mathbf{r} = (\sin t, 2 + 2\cos t, 0)$

$$\int_c \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}\cdot \mathbf{r}'dt = \int_0^{2\pi} \left(-\sin^2 t + 2\cos t + 2\cos^2 t\right) dt =$$

$$= -2\pi + [2\sin t]_0^{2\pi} + \frac{3}{2}\int_0^{2\pi} (1 + \cos 2t)\, dt = -2\pi + 3\pi = \pi.$$