MATH 353 Handout #6

1. Evaluate $\oint_{\mathcal{C}} x^2y^2\ dx + 4xy^3\ dy$ where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,3)$ and $(0,3)$, oriented positively.

   This is a good candidate for Green's Theorem. The integral is equivalent to the double integral $\int\int_D 4y^3 - 2x^2y dA$ where D is the solid triangle with the same vertices. That is, $D = \{(x,y)|0 \le x \le 1; 3x \le y \le 3\}$. So $\oint_{\mathcal{C}} x^2y^2\ dx + 4xy^3\ dy = \int_0^1 \int_{3x}^3 4y^3 - 2x^2y\ dy\ dx = \int_0^1 (y^4 - x^2y^2|_{3x}^3\ dx = \int_0^1 81 - 9x^2 - 72x^4\ dx = 81x - 3x^3 - (72/5)x^5|_0^1 = 78 - (72/5)$.

2. Evaluate $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle$ and $\mathcal{C}$ consists of the arc of the curve $y = \sin x$ from $(0,0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0,0)$.

   Another good candidate for Green's theorem (sorry, I forgot to scramble them). The region is $D = \{(x,y)|0 \le y \le \sin x;\ 0 \le x \le 1\}$. So $\int_0^\pi \int_0^{\sin x} 2x - 3y^2\ dy\ dx = \int_0^\pi 2xy - y^3|_0^{\sin x}\ dx = \int_0^\pi 2x \sin x - \sin^3 x\ dx = \int_0^\pi 2x \sin x - \int_0^\pi \sin^3 x\ dx$. The first integral is by parts: let $u = 2x$ and $dv = \sin x\ dx$. Then $\int_0^\pi 2x \sin x dx = -2x \cos x + 2 \sin x|_0^\pi = -2\pi \cos \pi + 2 \sin \pi = 2\pi$. The second integral is by substitution. Write $\sin^3 x = \sin x(1 - \cos^2 x)$ and let $u = \cos x$. Then $\int_0^\pi \sin^3 x\ dx = \int_0^\pi \sin x(1 - \cos^2 x)\ dx = -\cos x - (1/3)\cos^3 x|_0^\pi = -\cos \pi - 1/3 \cos^3 \pi + 1 + 1/3 = 8/3$. Together, the integral is $2\pi - 8/3$. Now, since the curve $\mathcal{C}$ is oriented CLOCKWISE, we have $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = 8/3 - 2\pi$.

3. Evaluate $\int \int_{\mathcal{S}} curl \mathbf{F} \bullet d\mathbf{S}$ where $\mathbf{F}(x,y,z) = yz, xz, xy$ and $\mathcal{S}$ is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

   This is a good candidate for Stokes's Theorem. The integral is equal to $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ where $\mathcal{C}$ is the curve of intersection of $z = 9 - x^2 - y^2$ and $z = 5$. That is, $\mathcal{C}$ is the curve given by $5 = 9 - x^2 - y^2$ or $x^2 + y^2 = 4$ and $z = 5$. So a good parametrization is given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), 5 \rangle$ where $0 \le t \le 2\pi$. So $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \langle yz, xz, xy \rangle \bullet (d\mathbf{r}/dt) dt = \int_0^{2\pi} \langle 5 \sin t, 5 \cos t, \sin t \cos t \rangle \bullet \langle -\sin t, \cos t, 0 \rangle dt = \int_0^{2\pi} 5(\cos^2 t - \sin^2 t)\ dt = 5 \int_0^{2\pi} \cos 2t dt = (5/2) \sin 2t|_0^{2\pi} = 0$.

4. Evaluate $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}(e^{-x}, e^x, e^z)$ and $\mathcal{C}$ is the boundary of the part of the plane $2x+y+2z = 2$ in the first octant, oriented counterclockwise when viewed from above.

   Another good candidate for Stokes's Theoerem, used in the OTHER direction. Calculate $curl(\mathbf{F})$ as the determinant of

$$\begin{bmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-x} & e^x & e^z \end{bmatrix}$$

   which is $\langle 0, 0, e^x\rangle$. The surface $\mathcal{S}$ that we will use is the part of the plane $2x+y+2z = 2$ in the first octant. A normal vector to this plane is $\langle 2, 1, 2\rangle$ and a normal vector is $N = \langle 2/3, 1/3, 2/3\rangle$. So $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} curl \bullet dS = \iint_{D} \langle 0, 0, e^x\rangle \bullet \langle 2/3, 1/3, 2/3\rangle dA$ where D is the shadow of S in the xy-plane. This is the triangle formed by the x-axis, the y-axis and the intersection of the plane $2x+y+2z = 2$ with $z = 0$, that is $y = 2-2x$. So the integral is $\int_0^1 e^x \int_0^{2-2x} \, dy \, dx = \int_0^1 2e^x - 2xe^x \, dx$. Now remember that integration by parts tells you that the antiderivative of xe^x is $xe^x - e^x + C$. So the integral is $2e^x - 2xe^x + 2e^x|_0^1 = 2e - 2e + 2e - 2 + 0 - 2 = 2e - 4$.

5. Calculate the flux of $\mathbf{F}(x, y, z) = \langle 4x^3z, 4y^3z, 3z^4\rangle$ out of the sphere $\mathcal{S}$ with radius R centered at the origin.

   The divergence of $\mathbf{F}$ is $div(\mathbf{F}) = 12x^2z + 12y^2z + 12z^3 = 12z(x^2 + y^2 + z^2) = 12zR^2$ for any (x, y, z) on the sphere. The Divergence Theorem says that the flux is equal to $\iiint_E 12zR^2 dV$ where E is the solid ball of radius R centered at the origin. Using spherical coordinates, the integral becomes

$$12R^2 \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho \cos(\phi) \rho^2 \sin(\phi) \; d\rho \; d\phi \; d\theta$$

$$= 3R^2 \int_0^{2\pi} \int_0^{\pi} \rho^4|_0^R \cos(\phi) \sin(\phi) \; d\phi \; d\theta$$

$$= 3\pi R^6 \sin^2(\phi)|_0^{\pi} = 0$$

6. Evaluate $\int_{\mathcal{C}} \mathbf{F} \bullet N ds$ where $\mathbf{F}(x, y) = \langle -y, x \rangle$ and $\mathcal{C}$ is the unit circle, oriented positively.

   Again, this is a good candidate for the Divergence Theorem, but the 2-D version. Calculate $div(\mathbf{F}) = 0$ and we immediately get $\int_{\mathcal{C}} \mathbf{F} \bullet N\ ds = \int\int_D 0\ dA = 0$.