1. Consider the domain T bounded by the triangle having vertices $(0,0)$, $(\pi,0)$, $(0,\pi)$. Evaluate

$$\int\int_T \frac{\sin y}{\pi - y}\ dA\ .$$

$$\int_0^{\pi}\int_0^{\pi-y} \frac{\sin y}{\pi - y}\ dx\ dy = \int_0^{\pi} \sin y\ dy = -\cos\pi + \cos 0 = 2$$

2. An open box has volume 40 cubic metres. The material for the front and bottom faces is 4 times as expensive as the material for the other three faces. Determine the dimensions that will minimize the cost of the box.

   There is obviously more than one way to do this one, we will use Lagrange multipliers.

   Let's say the box has width x, length y and height z. Then the volume is $V = xyz = 40$. The cost of manufacturing the box is $C(x,y,z) = 4(xz) + 4(xy) + 2(yz) + xz = 5xz + 4xy + 2yz$. Then $\nabla C = \langle 5z+4y, 4x+2z, 5x+2y\rangle$ and $\nabla V = \langle yz, xz, xy\rangle$. Setting $\nabla C = \lambda \nabla V$ we get 3 equations:

$$5z + 4y = \lambda yz$$
$$4x + 2z = \lambda xz$$
$$5x + 2y = \lambda xy$$

   Multiplying the first by x, the second by y and the third by z the right hand sides are all equal and we get

$$5xz + 4xy = 4xy + 2yz = 5xz + 2yz$$

   From the first two equations, we have $5xz = 2yz$ or $y = (5/2)x$. From the first and third expressions we get $4xy = 2yz$ or $z = 2x$. Plug these into the constraint to get $x(5/2)x(2x) = 40$ or $x^3 = 8$ so $x = 2$. Then $y = 5$ and $z = 4$. Since this is the only critical point, it must be the min. we are after (note that the larger the box gets, the more expensive it gets).

3. Use Lagrange multipliers to find the absolute minimum of the function $f(x,y) = 3 - \sqrt{2 - x^2 - y^2}$ on the sides of the triangle having vertices $(-1,0)$, $(1,2)$, $(1,-3)$.

   Finding the absolute min. of $3 - \sqrt{2 - x^2 - y^2}$ is the same as finding the absolute max. of $\sqrt{2 - x^2 - y^2}$ which is the same as finding the absolute max of $2 - x^2 - y^2$ which in turn is the same as finding the absolute min. of $x^2 + y^2$. Geometrically this means finding the point on the sides of the triangle closest to the origin.

   So we only need to check the portion of the triangle which is closest to the origin. Hence, we will use the method of Lagrange multipliers on the portion of the triangle where $y = x + 1$. Let $g(x,y) = y - x = 1$ and

$F(x,y) = x^2 + y^2$ (as above it is enough to minimize F). Then setting $\nabla F = \lambda \nabla g$ we obtain the equations

$$2x = -\lambda$$

$$2y = \lambda$$

Together, these imply that $x = -y$. Using this in the constraint equation $y - x = 1$, we have $y + y = 1$ so that $y = 1/2$ and $x = -1/2$. Since this is the only critical point, it must be the absolute maximum. Then $f(-1/2, 1/2) = 3 - \sqrt{2 - 1/4 - 1/4} = 3 - \sqrt{3/2}$.

4. Write down an integral which expresses the volume of the region in the first octant between the planes $x = 0$ and $x = 2y$, and below the parabolic cylinder $z = 1 - y^2$. Do not evaluate the integral.

   A good idea is to look at what is going on in the xy-plane. The portion of this solid in the xy-plane is the triangle bounded by the lines $x = 0$, $x = 2y$ and $y = 1$ (we get this from $z = 1 - y^2$ by setting $z = 0$). Thus, the region is $\{(x,y,z) \mid 0 \le x \le 2y; 0 \le y \le 1; 0 \le z \le 1 - y^2\}$. The integral is

$$\int_0^1 \int_0^{2y} \int_0^{1-y^2} 1 \; dz \; dx \; dy$$

5. Let R be the region lying between the paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$. Find its volume.

   Use cylindrical coordinates. The intersection of these two paraboloids is the circle $x^2 + y^2 = 2$, so this is the boundary of the domain in the xy-plane. Thus the integral is

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r \; dz \; dr \; d\theta = 2\pi \int_0^{\sqrt{2}} r(4 - 2r^2) \; dr$$
$$= 2\pi(2r^2 - (1/2)r^4|_0^{\sqrt{2}}) = 4\pi$$

6. It is found that a certain double integral will be simpler with the change of variables

$$\left\{ \begin{array}{c} x = r\sec\theta \\ y = r\tan\theta \end{array} \right\}.$$

   Determine dA in the new r, θ coordinates.

   That is, compute the determinant of

$$\begin{pmatrix} \sec\theta & r\sec\theta\tan\theta \\ \tan\theta & r\sec^2\theta \end{pmatrix}$$

   Which is $r\sec^3\theta - r\sec\theta\tan^2\theta = r\sec\theta$. So $dA = r\sec\theta \; dr \; d\theta$.

7. Answer "True" or "False" for each of the following. Do not write "T" or "F". In all these questions f is assumed to be a differentiable function.

(a) If

$$Hf(a,b,c) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5/3 \end{pmatrix}$$

is the Hessian matrix of the function $f(x,y,z)$ evaluated at the critical point $P = (a,b,c)$, then f has a local minimum at P. False (it has a local maximum at P)

(b) If P is the only interior critical point of a function f defined on a closed and bounded domain D, and if P is a local minimum, then P must be an absolute minimum. True

(c) Consider f restricted to the domain given by the curve $E = \{(x,y) : x^3 + y^3 = 1\}$. If f has a local minimum at $P = (0,1)$ and a local minimum at $Q = (1,0)$, then f must have an absolute minimum on the portion of E between P and Q. False (it must have a local max there)

(d) The subset $D \subset \mathbb{R}^2$ given by $D = \{(r,\theta) | 0 \le r \le \cos\theta,\ 0 \le \theta \le \pi/2\}$ is closed. True

(e) A subset $D \subseteq \mathbb{R}^3$ can be both open and closed. True (e.g.the empty set, or $\mathbb{R}^3$ itself)

(f) $\int_0^{2\pi} \int_0^2 \int_0^1 r\ dz\ dr\ d\theta$ is the volume of a right circular cylinder of height 1 and radius 2. True

(g) If f is defined on the rectangle $R = \{(x,y) \mid a \le x \le b; c \le y \le d\}$ then $\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$. True

(h) If g, h are integrable on a subset $D \subseteq \mathbb{R}^2$, and $g(x,y) > h(x,y)$ for all points (x,y) in D, then $\int\int_D g\, dA > \int\int_D h\, dA$. False, $\ge$ would be true since both could equal 0).

(i) $\displaystyle\int_0^2 \int_{x^2-4}^{4-x^2} (x^2 \cos y - xy^4) dydx = 2 \int_0^2 \int_0^{4-x^2} (x^2 \cos y - xy^4) dydx$

. True, because the integrand is an even function of y.

(j) Any point $(x,y) \in \mathbb{R}^2$ has unique polar coordinates (r,θ). False, it is true for every point except the origin.