MATH 353 FINAL HANDOUT-SOLUTION

For 1) $f(x,y)=x^2(y+1)^3+y^2$

$f_x=2x(y+1)^3=0 \qquad f_y=3x^2(y+1)^2+2y=0$

if $x=0$ then $y=0$;if $y=-1$ then No sol.

the only C.P. is $(0,0)$ Second.der.test

$f_{xx}=2(y+1)^3 \qquad A=2$

$f_{xy}=6x(y+1)^2 \qquad B=0$

$f_{yy}=6x^2(y+1)+2 \qquad C=2 \qquad D=-4$

$D<0, A>0$ loc. minimum

For abs. min. we have to show that $f(x,y)\geq f(0,0)=0$ for all (x,y)

BUT $f(R,-2)=-R^2+4<0$ for big R

thus it is NOT abs. min.

For 2)

$\int_{-1}^{0}(\int_{-1}^{\sqrt[3]{y}}\frac{dx}{x^4+1})dy$ given: $-1\leq y\leq 0 \qquad -1\leq x\leq \sqrt[3]{y}$

change the order $-1\leq x\leq 0 \qquad x^3\leq y\leq 0$

then

$$\int_{-1}^{0}(\int_{-1}^{\sqrt[3]{y}}\frac{dx}{x^4+1})dy= \int_{-1}^{0}(\frac{1}{x^4+1}\int_{x^3}^{0}dy)dx=$$

$$=\int_{-1}^{0}\frac{-x^3dx}{x^4+1}=\frac{1}{4}\left[-\ln(x^4+1)\right]_{-1}^{0}=\frac{\ln 2}{4}.$$

For 3)

the distance to $P(1,2,2)$. is $\sqrt{(x-1)^2+(y-2)^2+(z-2)^2}$

so looking for minimum of

$f(x,y,z)=(x-1)^2+(y-2)^2+(z-2)^2$ with $g(x,y,z)=x^2+y^2+z^2=36$

$\nabla f=\lambda\nabla g \qquad 2(x-1)=\lambda 2x \qquad 2(y-2)=\lambda 2y \qquad 2(z-2)=\lambda 2z$

gives $\lambda=\frac{x-1}{x}=\frac{y-2}{y}=\frac{z-2}{z}$ since $\lambda xyz\neq 0$

thus from $\frac{y-2}{y}=\frac{z-2}{z}\rightarrow z=y$

and from $\frac{x-1}{x}=\frac{y-2}{y}\rightarrow xy-y=yx-2x \qquad y=2x$

back to the sphere

$x^2+4x^2+4x^2=9x^2=36 \qquad x=\pm 2$ and we have 2 C.P.

for $(-2,-4,-4)$ the distance is $\sqrt{9+36+36}=9$ and for

$(2,4,4)$ is t**he closest poin**t since distance is $\sqrt{1+4+4}=3$ (minimum)

For 4 a)

$D=\{\ x+y\geq 2$ and $x^2+y^2\leq 4\}=\{0\leq x\leq 2, 2-x\leq y\leq\sqrt{4-x^2}\}$

so

$$\iint_D\frac{x+y}{x^2+y^2}dxdy=\int_0^2\left(\int_{2-x}^{\sqrt{4-x^2}}\frac{x+y}{x^2+y^2}dy\right)dx = \text{OR} = \int_0^2\left(\int_{2-y}^{\sqrt{4-y^2}}\frac{x+y}{x^2+y^2}dx\right)dy$$

$$= \int_0^2 \left(\int_{2-x}^{\sqrt{4-x^2}} \frac{2y}{x^2+y^2} dy \right) dx = \int_0^2 \left[\ln(x^2+y^2)\right]_{2-x}^{\sqrt{4-x^2}} dx = 2\ln 4 - \int_0^2 \ln(2x^2-4x+4)dx$$

by parts and partial fractions..

for b)

$D^* = \{r(\cos\theta + \sin\theta) \geq 2 \text{ and } 0 \leq r \leq 2, \theta \in \left[0, \frac{\pi}{2}\right]\}$

$$\iint_D \frac{x+y}{x^2+y^2} dxdy = \iint_{D^*} \frac{r(\cos\theta+\sin\theta)}{r^2} rdrd\theta = \int_0^{\frac{\pi}{2}} (\cos\theta+\sin\theta) \left(\int_{\frac{2}{\cos\theta+\sin\theta}}^{2} dr \right) d\theta =$$

$$= \int_0^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\left(2 - \frac{2}{\cos\theta+\sin\theta}\right) d\theta = \int_0^{\frac{\pi}{2}} 2(\cos\theta+\sin\theta)\, d\theta - 2\cdot\frac{\pi}{2} =$$

$$= 2\left[\sin\theta - \cos\theta\right]_0^{\frac{\pi}{2}} - \pi = 4 - \pi.$$

For 5)

the set B is a layer of a full paraboloid

use cylindrical coordinates

$B^* = \{1 \leq z \leq 4 \text{ and } z \geq r^2, \theta \in [0, 2\pi]\}$

so $\displaystyle \iiint_B \frac{zdV}{\sqrt{x^2+y^2}} = \iiint_{B^*} \frac{z}{\sqrt{r^2}} rdrdzd\theta = 2\pi \iint_{D^*} zdrdz$

where $D^* = \{1 \leq z \leq 4 \text{ ,} 0 \leq r \leq \sqrt{z}\}$

the integral $\displaystyle I = 2\pi \int_1^4 z\left(\int_0^{\sqrt{z}} dr\right)dz = 2\pi \int_1^4 z^{\frac{3}{2}}dz = \frac{4\pi}{5}\left[z^{\frac{5}{2}}\right]_1^4 = \frac{4\pi}{5}\left[2^5 - 1\right] = \frac{124\pi}{5}.$

For 6)

$S = \{\ z = \sqrt{3x^2+3y^2} \quad x + z \leq 4\}$

so $0 \leq \sqrt{3x^2+3y^2} \leq 4 - x \quad \rightarrow \quad D = ?$

$\displaystyle SA = \iint_S dS = \iint_D \|\mathbf{n}\|\, dxdy$

$$\mathbf{n} = (\nabla z, -1) = \left(\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}, \frac{\sqrt{3}y}{\sqrt{x^2+y^2}}, -1 \right) \text{or} \left(\frac{3x}{\sqrt{3x^2+3y^2}}, \frac{3y}{\sqrt{3x^2+3y^2}}, -1 \right)$$

and $\displaystyle \|\mathbf{n}\|^2 = \frac{3x^2+3y^2}{x^2+y^2} + 1 = 4 \qquad \|\mathbf{n}\| = 2$

we have to find D the intersection of the given cone and plane

$4 - x \geq \sqrt{3x^2+3y^2} \qquad \text{for } x \leq 4 \qquad 16 - 8x + x^2 = 3x^2 + 3y^2$

$16 = 2x^2 + 8x + 3y^2 \qquad 24 = 2(x+2)^2 + 3y^2$

finally $\displaystyle D = \{2(x+2)^2 + 3y^2 \leq 24\} = \{\left(\frac{x+2}{\sqrt{12}}\right)^2 + \left(\frac{y}{\sqrt{8}}\right)^2 \leq 1\}$

so

$SA = 2 \cdot \mathit{area}$ of an ellipse $= 2\pi ab = 2\pi\sqrt{12}\sqrt{8} = 8\pi\sqrt{6}.$

For 7)

the curve is closed,oriented clockwise, $n = 2$ use Green's theorem

$\displaystyle \oint_c \mathbf{F} \cdot \mathbf{ds} = -\iint_T \left[(F_2)_x - (F_1)_y\right] dxdy$ because of the orientation of c

where $T = \{0 \leq x \leq 2, 2 - x \leq y \leq 2\} = \{0 \leq y \leq 2, 2 - y \leq x \leq 2\}$

and $\left[(F_2)_x - (F_1)_y\right] = \pi\cos\pi x - 3xy^2$

$$= -\iint_T [\pi\cos\pi x - 3xy^2]\,dxdy = \int_0^2 3y^2(\int_{2-y}^2 xdx)dy - \int_0^2 \pi\cos\pi x(\int_{2-x}^2 dy)dx =$$

$$= \int_0^2 3y^2\cdot\frac{2^2-(2-y)^2}{2}dy - \int_0^2 x\cdot\pi\cos\pi x dx = \frac{3}{2}\int_0^2 (4y^3-y^4)\,dy - [x\sin\pi x]_0^2 - \left[\frac{\cos\pi x}{\pi}\right]_0^2 =$$

$$= \frac{3}{2}\left[2^4 - \frac{2^5}{5}\right] - 0 = \frac{72}{5}.$$

For 8)

by Product Rule

$div(\phi\mathbf{F}) = (\phi F_1)_x + (\phi F_2)_y = \phi(F_1)_x + \phi_x F_1 + \phi(F_2)_y + \phi_y F_2 =$

$= \phi(F_1)_x + \phi(F_2)_y + (\phi_x,\phi_y)\bullet(F_1,F_2) = \phi div\mathbf{F} + grad\phi\bullet\mathbf{F}$

.**For 9)**

since c is closed ,$n=3$.we can use Stokes's Theorem

$$I = \oint_c \mathbf{F}\cdot\mathbf{ds} = \iint_S curl\ \mathbf{F}\cdot\mathbf{dS} \qquad \mathbf{S} = ?$$

where

$$curl\ \mathbf{F} = \left\|\begin{matrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ x^2+y & y^3-x & z^4 \end{matrix}\right\| = (0,0,-2)$$

and

S is a part of the plane $z = 2-2x+3y$ so $\mathbf{n} = (-\nabla z, 1) = (2,-3,1)$

inside the cylinder $\{x^2+y^2\le 4\} = D \qquad curl\ \mathbf{F}\cdot\mathbf{n} = -2$

thus $\quad I = \iint_D (-2)\,dxdy = -2\cdot$area of $D = -8\pi.$

For 10)

since S is a closed surface use Divergence theorem $div\mathbf{F} = 2x+2y+2z$

and

$$I = \iint_S \mathbf{F}\cdot\mathbf{dS} = 2\iiint_B (x+y+z)\,dxdydz \text{ where } B = \{x^2+y^2+4(z-1)^2\le 4\}$$

ellipsoid symmetrical in x and in y so $I = 0+0+2\iiint_B zdxdydz =$

projection on the z-axis :mz$\in[0,2]$ so $\qquad I = 2\int_0^2 z\iint_{D_z} dxdy$

where $D_z = \{x^2+y^2\le 4-4(z-1)^2 = 8z-4z^2\}$ circular disc for each fixed z with area $= \pi\,(8z-4z^2)$

thus $\quad I = 2\int_0^2 z\pi\,[8z-4z^2]\,dz = 8\pi\int_0^2(2z^2-z^3)dz = 8\pi\left[\frac{2}{3}\cdot 2^3 - \frac{2^4}{4}\right] = \frac{32}{3}\pi$

OR cylindrical coordinates $2\iiint_B zdxdydz = 4\pi\iint_{D^*} zrdrdz$

where $D^* = \{r^2+4\,(z-1)^2\le 4\}$

the integral $= 4\pi \int\limits_0^2 z \left(\int\limits_0^{\sqrt{4-4(z-1)^2}} r dr \right) dz = 8\pi \int\limits_0^2 z\left(2z - z^2\right) dz = ...$

Or modified spherical coord. $z = 1 + 2\rho\cos\phi$

ALSO

$I = 2\iiint\limits_B z dxdydz = 2\iiint\limits_B (z-1) dxdydz + 2$volume of

the ellipsoid: $B = \{x^2 + y^2 + 4(z-1)^2 \le 4\} = \left\{ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + (z-1)^2 \le 1 \right\}$

$= 0 + 2 \cdot \frac{4}{3}\pi abc = \frac{32}{3}\pi.$

For 11)

the curve is not closed but the field is conservative

so we can find a potential $F = \nabla f$

$f_x = y \qquad f_y = x \qquad f_z = 2z$

$f(x, y, z) = xy + z^2$ and $\int\limits_c \mathbf{F} \bullet \mathbf{ds} = f(B) - f(A) = 5 - 3 = 2$

OR find a parametrization of c

from the cylinder $x = \sqrt{2}\cos t,\ y = \sqrt{2}\sin t$ and from $z = 2xy$

we have $\mathbf{r}(t) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 4\cos t \sin t\right) = \left(\sqrt{2}\cos t, \sqrt{2}\sin t, 2\sin 2t\right)$

now for A $t = \frac{3}{4}\pi$, and for B $t = \frac{\pi}{4}$

$\mathbf{r}'(t) = \left(-\sqrt{2}\sin t, \sqrt{2}\cos t, 4\cos 2t\right)$ and the field on c

$\mathbf{F} = (y, x, 2z)_c = \left(\sqrt{2}\sin t, \sqrt{2}\cos t, 4\sin 2t\right)$ then

$\mathbf{F} \cdot \mathbf{r}' = -2\sin^2 t + 2\cos^2 t + 8\sin 4t$

$\int\limits_c \mathbf{F} \cdot \mathbf{ds} = \int\limits_{\frac{3}{4}\pi}^{\frac{\pi}{4}} \mathbf{F} \cdot \mathbf{r}' dt = -\int\limits_{\frac{1}{4}\pi}^{\frac{3}{4}\pi} \left[2\cos 2t + 8\sin 4t\right] dt =$

$= \left[-\sin 2t + 2\cos 4t\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 1 + 1 - 2 + 2 = 2.$

For 12)

$\mathbf{F}(x, y, z) = \left(\arctan z(x^2 + y^2), \ln(1 + y^2 + z^2), y\ e^{xyz}\right)$

for any (x, y, z)

$div\mathbf{F} = \frac{2xz}{1 + z^2\left(x^2 + y^2\right)^2} + \frac{2y}{1 + y^2 + z^2} + xy^2\ e^{xyz}$

and $curl\mathbf{F} = \begin{bmatrix} + & - & + \\ \partial_x & \partial_y & \partial_z \\ \arctan z(x^2 + y^2) & \ln(1 + y^2 + z^2) & ye^{xyz} \end{bmatrix} =$

$= \left(e^{xyz}\left(1 + xyz\right) - \frac{2z}{1 + y^2 + z^2}, -y^2 z e^{xyz} + \frac{x^2 + y^2}{1 + z^2\left(x^2 + y^2\right)^2}, -\frac{2yz}{1 + z^2\left(x^2 + y^2\right)^2} \right).$