MATH 353 Handout #1T Solution

A

For 1a)

the set $S=\left\{(x,y);\frac{|x|}{|y|}\le 1\right\}$ is symmetrical in x and y,it means $\pm x,\pm y$ are in if $y\ne 0$,
let's investigate for $x\ge 0, y>0$....1st quadrant $x\le y$ points are above the line $y=x$,
now reflection in both axes gives us the set in other quadrants,the origin is excluded
we can see that all points on both lines $y=x$ and $y=-x$
are the boundary points since any neighbourhood contains points from the set
and also outside the set.Thus $\partial S=\{(x,y);y=x \text{ and } y=-x\}$
all points on the boundary except (0,0) are in the set,but not all
so the set is **NOT closed**,some are in so the set can**not be open**.
tthe set is **NOT bounded**

For 1b)

the set $S=\{(x,y);y-2x=1,1\le y\le 3\}$is a line segment
with the ends $A(0,1)$ and $B(1,3)$
any point on the line segment is a boundary point
since close by are points from the segment and also from the complement R^2-S
$\partial\ S=S$ and the set is **closed and bounded**

For 1c) $S=\{$ all irrational numbers between 0 and 1$\}\subset R$
the boundary is the closed interval $[0,1]$ bigger that the set $S\subset\partial S$
since in any small interval we can find both rational and irrational numbers
but $S\ne\partial S$ so the set is **neither closed nor open,**but **bounded.**

For 2)

the function $f(x,y)=xye^{-2x^2-\frac{y^4}{4}}$ is differentiable everywhere,
for critical points solve $\nabla f=\mathbf{0}$
$f_x=ye^{-2x^2-\frac{y^4}{4}}(1-4x^2)=0$.......$y=0$ or $x=\pm\frac{1}{2}$
$f_y=xe^{-2x^2-\frac{y^4}{4}}(1-y^4)=0$......$x=0$ or $y=\pm 1$
so all possible combinations are :$(0,0)$,$\left(\pm\frac{1}{2},1\right)$, $\left(\pm\frac{1}{2},-1\right)$ ......5 critical points
$f(0,0)=0$ is neither max nor min since the values of f are positive if $xy>0$,
and negative if $xy<0$ for (x,y) close to $(0,0)$
$f(\frac{1}{2},1)=f(-\frac{1}{2},-1)=\frac{1}{2}e^{-\frac{3}{4}}$ and $f(\frac{1}{2},-1)=f(-\frac{1}{2},1)=-\frac{1}{2}e^{-\frac{3}{4}}$
so first two could be maxima ,the latter two minima points
Second Der.Test:
$f_{xx}=ye^{-2x^2-\frac{y^4}{4}}(-12x+16x^3)=4xye^{-2x^2-\frac{y^4}{4}}(-3+4x^2)$......$A$
$f_{xy}=e^{-2x^2-\frac{y^4}{4}}(1-4x^2)(1-y^4)$................................B
$f_{yy}=xe^{-2x^2-\frac{y^4}{4}}(-5y^3+y^7)=xy^3e^{-2x^2-\frac{y^4}{4}}(-5+y^4)$..........$C$
the discriminant $D=B^2-AC$
at $(0,0)$.....$D=1>0$...........**saddle point**
at $\left(\frac{1}{2},1\right)$ and $\left(-\frac{1}{2},-1\right)$ ....$B=0,A<0,C<0$ so $D<0$.......**local maxima**
at $\left(\frac{1}{2},-1\right)$ and $\left(-\frac{1}{2},1\right)$ ..$B=0,A>0,C>0$ so $D<0$.......**local minima**

B

For 3a) in $S=\{(x,y);\ln(xy)\le 0\}$

ln is defined only for positive numbers so $xy > 0$,1st and 3rd quadrants
to solve $\ln(xy) \leq 0$ apply exp. function to both sides
so $xy \leq 1$, together $0 < xy \leq 1$ below or above hyperbola $y = \frac{1}{x}$
thus the boundary $\partial S = \{x = 0 \text{ or } y = 0 \text{ or } xy = 1\}$both axes and hyperbola
part is in (hyperbola),part is out (axes)
so the set S is **neither open nor closed,not bounded.**

For 3b)
we can see that the set $S = \{(x, y); 0 < x^2 + y^2 < 4\}$is a circular disk without the center and without the circle
Thus $\partial S = \{(x, y); x^2 + y^2 = 4 \text{ and } (0, 0)\}$,
the whole boundary is outside the set ,so S is **open,and bounded.**

For 3c) the set $S = \left\{ \frac{n}{3n+1} \right\}_{n=1}^{\infty} \subset R$
is a sequence convergent to $\frac{1}{3}$ so $\partial S = S \cup \left\{\frac{1}{3}\right\}$ but the limit is not included
so the set is **neither open nor closed,but bounded**

For 4)
the function $f(x, y) = 2xy^2 - x^2y + 4xy$ is differentiable everywhere,
for critical points solve
$f_x = 2y^2 - 2xy + 4y = 2y(y - x + 2) = 0.......y = 0 \text{ or } x - y = 2$
$f_y = 4xy - x^2 + 4x = x(4y - x + 4) = 0..... \quad x = 0 \text{ or } x - 4y = 4$
all combinations :$x = y = 0$ then $y = 0$ and from the second one $x = 4$,
then $x = 0$ and from the first one $y = -2$;
finally solve the system $x - y = 2$, $x - 4y = 4$
we got 4 critical points : $(0, 0)...(0, -2) .. (4, 0) ..(\frac{4}{3}, -\frac{2}{3})$
for Second Derivative Test
$f_{xx} = -2y..A \qquad f_{xy} = 4y - 2x + 4...B \qquad f_{yy} = 4x...C$
disc. $D = B^2 - AC$

points	$(0,0)$	$(0,-2)$	$(4,0)$	$\left(\frac{4}{3}, \frac{-2}{3}\right)$
A	0	4	0	$\frac{4}{3}$
B	4	-4	-4	$\frac{-4}{3}$
C	0	0	16	$\frac{16}{3}$
D	16	16	16	$\frac{-16}{3}$

Therefore first 3 are **saddle points** since $D > 0$
and f has only one **local minimum** at $(\frac{4}{3}, -\frac{2}{3})$ since $D < 0$ and $A > 0$.

C

For 5a)
in $S = \left\{(x, y); \frac{x^2}{y} \geq 1\right\}$the fraction is defined only for $y \neq 0$,
also we can see that y must be positive and $x^2 \geq y$
together $x^2 \geq y > 0$
We can see that the boundary $\partial S = \{\ y = 0 \text{ or } y = x^2\}$ ....x- axis and parabola
part is in (parabola) ,part is out (axis) so the set S is **neither open nor closed,unbounded.**

For 5b).
the set $S = \{(x, y, z); x^2 + y^2 + 2z^2 = 4\}$ is an elliptical shell or ellipsoid
$\partial S = S$, since any point on the shell has "close by" some points on the shell
and some points outside and inside the shell

the whole boundary is part of the set ,so S is **closed, bounded.**
(inside big ball with radius 4).
For 6a)
the function $f(x,y) = xy\,(4-x-4y)$is differentiable everywhere,for critical points solve
$f_x = y\,(4-x-4y-x) = 2y\,(2-2y-x) = 0$.......$y = 0$ or $x+2y = 2$
$f_y = x\,(4-x-4y-4y) = x\,(4-x-8y) = 0$......$x = 0$ or $x+8y = 4$
so all possible combinations are :
$(0,0),(0,1),(4,0)$and $\left(\frac{4}{3},\frac{1}{3}\right)$ ..(by solving the system) 4 critical points
to classify the critical points use the second derivative test:
$f_{xx} = -2y$ A
$f_{xy} = 2\,(2-2y-x-2y) = 2\,(2-x-4y)$ B
$f_{yy} = -8x$ C
the discriminant $D = B^2 - AC$

points	$(0,0)$	$(0,1)$	$(4,0)$	$\left(\frac{4}{3},\frac{1}{3}\right)$
A	0	-2	0	$\frac{-2}{3}$
B	4	-4	-4	$\frac{-4}{3}$
C	0	0	-32	$\frac{-32}{3}$
D	16	16	16	$\frac{-16}{3}$

at $(0,0),(0,1),(4,0)$.....$D>0$ **saddle points**
at $\left(\frac{4}{3},\frac{1}{3}\right)$ $D<0, A<0$ **local maximum**
For 6b).
NO critical point from part a) is inside the triangle $\triangle\, ABC$
with vertices $A\,(0,0)\,, B\,(0,1)$ and $C\,(1,0)$ .so we have to investigate the boundary
$AC = \{y = 0, 0 \leq x \leq 1\}$ but $f = 0$;
$AB = \{x = 0, 0 \leq y \leq 1\}$ but again $f = 0$;
$BC = \{y = 1-x, 0 \leq x \leq 1\}$
and $f(x,1-x) = x(1-x)(4-x-4+4x) = 3(x^2-x^3) = g(x)$
for critical points on the line segment BC solve $g'(x) = 0$
$g'(x) = 3(2x-3x^2) = 3x(2-3x) = 0$ and $x = 0$ or $x = \frac{2}{3}$
therefore besides corners we have a point $\left(\frac{2}{3},\frac{1}{3}\right)$
and $f\left(\frac{2}{3},\frac{1}{3}\right) = g(\frac{2}{3}) = \frac{4}{9}$ is the **maximum value** ;
at all corners and on the line segments AB, AC the value is 0 is the**.minimum value.**
D
For 7a)
in $S = \left\{(x,y); y \leq \dfrac{1}{x}\right\}$the function $\frac{1}{x}$ is defined for $x \neq 0$ so y-axis is excluded
for $x>0$ y is positive or negative under or on the hyperbola,
and for $x<0$ y is negative and under or on the hyperbola
.
we can see that the boundary $\partial S = \left\{x = 0 \text{ or } y = \dfrac{1}{x}\right\}$consists of y-axis and hyperbola
part of the boundary is in (hyperbola) ,part is out (y-axis)
therefore the set S is **neither open nor closed,unbounded.**
For 7b)
we can see that the set $S = \left\{(x,y); 9 < \dfrac{1}{x^2+y^2}\right\}$is a circular disk

without the center and without the circle: $0 < x^2 + y^2 < \frac{1}{3^2}$
Thus $\partial S = \left\{(x, y); x^2 + y^2 = \frac{1}{9}\right\} \cup \{\ (0,0)\}$
the whole boundary is outside the set ,so S is **open,bounded**

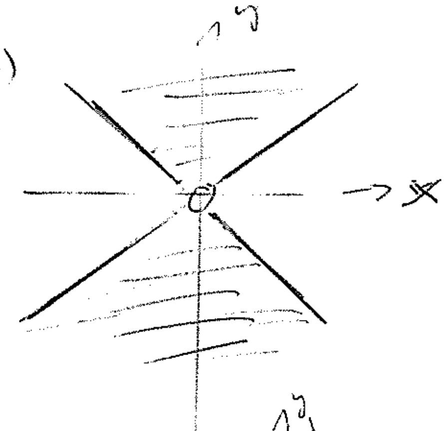
For 7c)
the set $S = \{\ \sqrt[n]{n}\}_{n=1}^{\infty} \subset R$ is a sequence convergent to $1 = L = a_1$
so $\partial S = S$ and the set is **closed and bounded**
since

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim\limits_{n\to\infty} \dfrac{\ln n}{n}} = e^0 = 1$$

(you can use L'Hop.Rule for the limit of the exponent.)
$\sqrt[1]{1} = 1, \sqrt{2} = 1.41 = \sqrt[4]{4}, \sqrt[3]{3} = 1.44, \sqrt[5]{5} = 1.379, ...\ \sqrt[10]{10} = 1.26....\ \sqrt[100]{100} = 1.047....$

For 8)
the function $f(x, y) = 3y^3 - x^2y + x^2$ is differentiable everywhere,for critical points solve
$f_x = -2xy + 2x = 2x(1 - y) = 0 \qquad y = 1$ or $x = 0$
$f_y = 9y^2 - x^2 = 0 \qquad x^2 = 9y^2 \qquad$ so if $y = 1$ then $x = \pm 3$ and if $x = 0$ also $y = 0$
$(0, 0), (\pm 3, 1)$ ......3 critical points $\qquad f(0, 0) = 0 \qquad f(\pm 3, 1) = 3$
$f_{xx} = 2 - 2y......A \qquad f_{xy} = -2x..........B \qquad f_{yy} = 18y..........C$
the discriminant $D = B^2 - AC$
at (0,0).....$D = 0$.......... NO conclusion from the TEST
at $(\pm 3, 1) \qquad A = 0, B = \mp 6, C = 18$ so $D = 36 > 0$.......**saddle points**
We have to go back to the origin and investigate the values at the points
around the origin , for example $\qquad f(0, y) = 3y^3$
so values are positive for $y > 0$ and negative for $y < 0$.........**saddle point** again.

1 a)



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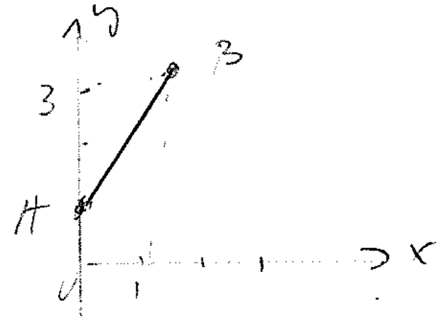

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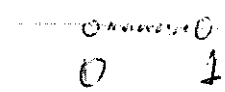
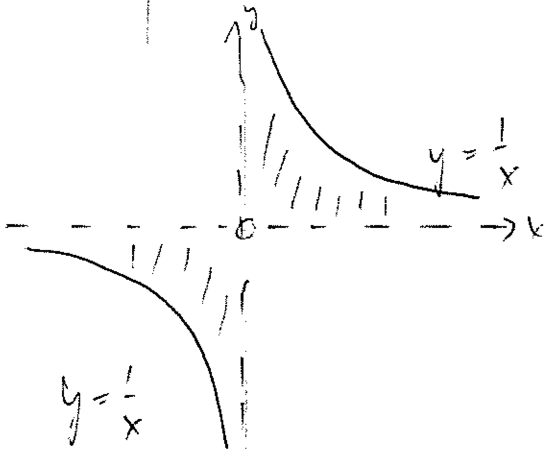


3 a)



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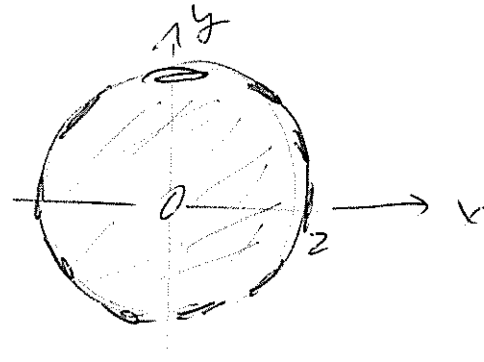
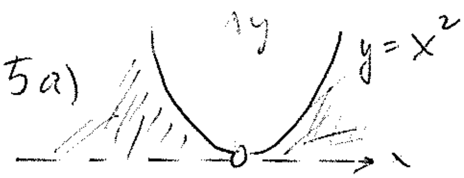
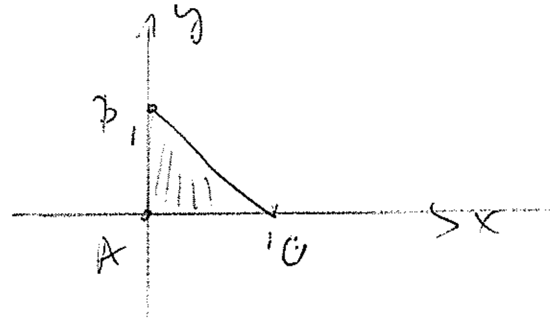


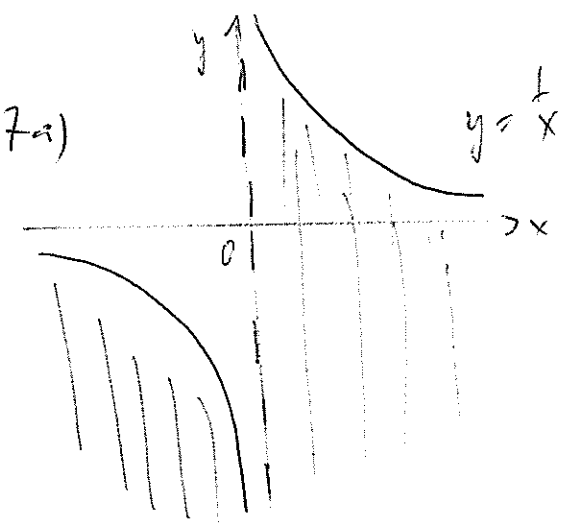
5 a)



6 b)



7 a)



7 b)

c)